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Note

Ball packings with high chromatic numbers from strongly regular graphs

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ABSTRACT

Inspired by Bondarenko's counter-example to Borsuk's conjecture, we notice some strongly regular graphs that provide examples of ball packings whose chromatic numbers are significantly higher than the dimensions. In particular, from generalized quadrangles we obtain unit ball packings in dimension $q^3 - q^2 + q$ with chromatic number $q^3 + 1$, where q is a prime power. This improves the previous lower bounds for the chromatic number of ball packings.

1. The problem and previous works

A *ball packing* in d-dimensional Euclidean space is a collection of balls with disjoint interiors. The *tangency graph* of a ball packing takes the balls as vertices and the tangent pairs as edges. The *chromatic number* of a ball packing is defined as the chromatic number of its tangency graph.

The Koebe–Andreev–Thurston disk packing theorem says that every planar graph is the tangency graph of a 2-dimensional ball packing. The following question is asked by Bagchi and Datta in [2] as a higher dimensional analogue of the four-color theorem:

Problem. What is the maximum chromatic number $\chi(d)$ over all the ball packings in dimension d?

The authors gave $d + 2 \leq \chi(d)$ as a lower bound since it is easy to construct $d + 2$ mutually tangent balls. By ordering the balls by size, the authors also argued that $\kappa(d) + 1$ is an upper bound, where $\kappa(d)$ is the kissing number for dimension d. For information, the current asymptotic bounds for $\kappa(d)$ are [21,29]

$$2^{0.2075\ldots n(1+o(1))} \leq \kappa(d) \leq 2^{0.401n(1+o(1))}.$$

However, the case of $d = 3$ has already been investigated by Maehara [25], who proved that $6 \leq \chi(3) \leq 13$. His construction for the lower bound uses a variation of Moser's spindle, which is the tangency graph of a *unit* disk packing in dimension 2 with chromatic number 4, and the following lemma:

Lemma. *If there is a unit ball packing in dimension d with chromatic number χ, then there is a ball packing in dimension $d + 1$ with chromatic number $\chi + 2$.*

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The technique of Maehara [25] can be easily generalized to higher dimensions and gives $d+3 \le \chi(d)$.

Another progress is made by Cantwell in an answer on MathOverflow [9], who proved that the graph of the halved 5-cube (also called the Clebsch graph) is the tangency graph of a 5-dimensional *unit* ball packing with chromatic number 8. Then Lemma implies that $10 \le \chi(6)$. This argument can be generalized to higher dimensions using a result of Linial, Meshulam and Tarsi [24], and gave $d+4 \le \chi(d)$ for $d = 2^k - 2$.

As we have seen, both constructions study the chromatic number of unit ball packings and invoke Lemma. We will do the same. A unit ball packing can be regarded as a set of points such that the minimum distance between pairs of points is at least 1, then the tangency graph of the packing is the unit-distance or minimum-distance graph for these points. By ordering the unit balls by height, we see that the chromatic number of a unit ball packing is at most one plus the one-side kissing number.

Recall that the *finite version* of the Borsuk conjecture can be formulated as follows: the chromatic number of the maximum-distance graph for a point set in $\mathbb{R}^d$ is at most $d+1$. So the chromatic number problem for unit ball packings is the "opposite" of the Borsuk conjecture. The Borsuk conjecture was first disproved by Kahn and Kalai [22]. Their result leads to a sub-exponential lower bound for the chromatic number of maximum-distance graphs. For comparison, if no bound is imposed to the distances, the chromatic number of general unit-distance graphs has an exponential lower bound [12].

Recently, Bondarenko [3] found a counter-example for Borsuk conjecture in dimension 65. His construction was then slightly improved by Jenrich [20] to dimension 64, which is the current record for the smallest counter-example. Their construction is based on geometric representations of strongly regular graphs. In this note, we use the technique of Bondarenko to find unit ball packings with strongly regular tangency graphs, whose chromatic numbers are significantly higher than their dimensions. In particular

Theorem. *For every prime power q, there is a unit ball packing of dimension $d = q^3 - q^2 + q$ whose tangency graph is strongly regular with chromatic number $\chi(d) = q^3 + 1$.*

Examples are given by the graphs of generalized quadrangles with parameters (q, q^2). This yields the first non-constant lower bound for the difference $\chi(d) - d$.

Remark 1. The current lower bound for $\chi(d)$ is linear and the upper bound is exponential. Improvements are encouraged for both asymptotic bounds. For information, the tangency graph of a ball packing in dimension d has a small clique number ($\le d+2$). On the other hand, there exist unit ball packings whose tangency graphs have large minimum degrees ($>2^{\sqrt{d}}$) [1].

2. Strongly regular graphs

We use [8] for general references on strongly regular graphs.

Let G be a strongly regular graph with parameters (v, k, λ, μ). That is, G is a k-regular graph on v vertices such that every pair of adjacent vertices has λ neighbors in common and every pair of non-adjacent vertices has μ neighbors in common. We assume that

$$\lambda - \mu \ge -2k/(v-1). \tag{1}$$

If this is not the case, we may replace G by its complement $\bar{G}$, which is a strongly regular graph with parameters $(v, v-k-1, v-2k-2+\mu, v-2k+\lambda)$. For our study of ball packings, we may focus on connected graphs, therefore $\mu > 0$. For any vertex of G, the graphs induced by its neighbors in G and by its neighbors in $\bar{G}$ are respectively the *first* and the *second subconstituent* of G.

The adjacency matrix A of G has three eigenvalues k, r, s with multiplicities $1, f, g$, respectively. They can be expressed in terms of the parameters as follows:

$$r, s = (\lambda - \mu \pm \delta)/2,$$
$$f, g = (v - 1 \pm \Delta)/2,$$

where $\delta = \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}$ and $\Delta = ((v-1)(\mu-\lambda) - 2k)/\delta \le 0$. The eigenvalues of $\bar{G}$ are $v-k-1, -s-1, -r-1$ with multiplicities $1, g, f$, respectively. Note that $r > 0 > s+1$ and $f \le g$.

Let I be the identity matrix and J the all-ones matrix. Then

$$E = (A - sI)(I - J/v)$$

is an eigenmatrix of A corresponding to the eigenvector r, and the column vectors of E (labeled by vertices of G) form a spherical 2-distance set on the sphere $\mathbb{S}^{f-1} \subset \mathbb{R}^f$, with angles $\cos\alpha = r/k$ for adjacent vertices and $\cos\beta = -(r+1)/(v-k-1)$ for non-adjacent vertices [3]; see also [6]. By putting a ball of radius $\sin(\alpha/2) = \sqrt{(1-r/k)/2}$ at each point of the 2-distance set, we obtain a ball packing whose tangency graph is G.

By Hoffman's bound [18] (see also [11] [6]), the clique number of the complement $\omega(\bar{G})$ is at most $1 + (v-k-1)/(1+r)$, so the chromatic number $\chi(G)$ is at least

$$v \Big/ \left(1 + \frac{v-k-1}{1+r}\right) = 1 - k/s.$$