Discrete Mathematics 340 (2017) 1649–1656

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Completely regular codes with different parameters giving the same distance-regular coset graphs

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ARTICLE INFO

Article history:
Received 2 December 2015
Received in revised form 27 January 2017
Accepted 1 March 2017

Keywords:
Bilinear forms graph
Completely regular code
Completely transitive code
Coset graph
Distance-regular graph
Distance-transitive graph
Kronecker product construction
Lifting of a field
Uniformly packed code

ABSTRACT

We construct several classes of completely regular codes with different parameters, but identical intersection array. Given a prime power q and any two natural numbers a, b, we construct completely transitive codes over different fields with covering radius $\rho = \min\{a, b\}$ and identical intersection array, specifically, one code over $\mathbb{F}_{q^r}$ for each divisor r of a or b. As a corollary, for any prime power q, we show that distance regular bilinear forms graphs can be obtained as coset graphs from several completely regular codes with different parameters.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of the order q and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A q-ary linear code C of length n is a k-dimensional subspace of $\mathbb{F}_q^n$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its *distance to the code* C is $d(\mathbf{v}, C) = \min_{\mathbf{x}\in C}\{d(\mathbf{v}, \mathbf{x})\}$, the *minimum distance* of the code is $d = \min_{\mathbf{v}\in C}\{d(\mathbf{v}, C\setminus\{\mathbf{v}\})\}$ and the *covering radius* of the code C is $\rho(C) = \max_{\mathbf{v}\in\mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. We say that C is a $[n, k, d; \rho]_q$ code. The *Hamming weight* $\mathrm{wt}(\mathbf{v})$ of a vector $\mathbf{v} \in \mathbb{F}_q^n$ wt() is the number of its nonzero entries, i.e. $\mathrm{wt}(\mathbf{v}) = d(\mathbf{v}, 0)$. Let $D = C + \mathbf{x}$ be a *coset* of C, where +means the component-wise addition in $\mathbb{F}_q$. The *weight* $\mathrm{wt}(D)$ of D is the minimum weight of the vectors in D. The *weight distribution* of D is the $(n + 1)$-tuple $(w_0, w_1, \dots, w_n)$ of nonnegative integers, where w_i is the number of codewords of D of weight i.

For a given q-ary code C with covering radius $\rho = \rho(C)$ define

$$C(i) = \{\mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i\}, \quad i = 1, 2, \dots, \rho.$$

Say that two vectors $\mathbf{x}$ and $\mathbf{y}$ are *neighbors* if $d(\mathbf{x}, \mathbf{y}) = 1$. For two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ over $\mathbb{F}_q$ denote by $\langle \mathbf{x}, \mathbf{y}\rangle$ their inner product over $\mathbb{F}_q$, i.e.

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + \cdots + x_ny_n.$$

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http://dx.doi.org/10.1016/j.disc.2017.03.001

The linear code $C^{\perp} = \{\mathbf{v} \mid \langle \mathbf{v}, \mathbf{x}\rangle = 0 \text{ for all } \mathbf{x} \in C\}$ is the *dual* code of C. Let $s(C)$ be the *outer distance* of C, i.e. the number of different nonzero weights of codewords in the dual code $C^{\perp}$.

Definition 1.1 ([4]). A q-ary code C with covering radius ρ is called *completely regular* if the weight distribution of any coset D of C is uniquely defined by the minimum weight of D.

An equivalent definition of completely regular codes is due to Neumaier [11].

Definition 1.2 ([11]). A q-ary code C is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number c_l of neighbors in $C(l-1)$ and the same number b_l of neighbors in $C(l+1)$. Define $a_l = (q-1)n - b_l - c_l$ and set $c_0 = b_\rho = 0$. Denote by $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ the intersection array of C.

Let M be a monomial matrix, i.e. a matrix with exactly one nonzero entry in each row and column. If q is a prime, then $\mathrm{Aut}(C)$ consist of all monomial $(n \times n)$-matrices M over $\mathbb{F}_q$ such that $\mathbf{c}M \in C$ for all $\mathbf{c} \in C$. If q is a power of a prime number, then $\mathrm{Aut}(C)$ also contains any field automorphism of $\mathbb{F}_q$ (which can be seen as maps of $\mathbb{F}_q^n$ into itself by acting on each of the coordinates) preserving C. The group $\mathrm{Aut}(C)$ acts on the set of cosets of C in the following way: for all $\sigma \in \mathrm{Aut}(C)$ and for every vector $\mathbf{v} \in \mathbb{F}_q^n$ we have $(\mathbf{v} + C)^\sigma = \mathbf{v}^\sigma + C$.

Definition 1.3 ([7,15]). Let C be a linear code over $\mathbb{F}_q$ with covering radius ρ. Then C is completely transitive if $\mathrm{Aut}(C)$ has $\rho + 1$ orbits in its action on the cosets of C.

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.

Definition 1.4 ([1]). Let C be a q-ary code of length n and let ρ be its covering radius. We say that C is *uniformly packed* in the wide sense if there exist rational numbers $\alpha_0, \ldots, \alpha_\rho$ such that for any $\mathbf{v} \in \mathbb{F}_q^n$

$$\sum_{k=0}^{\rho} \alpha_k f_k(\mathbf{v}) = 1, \tag{1}$$

where $f_k(\mathbf{v})$ is the number of codewords at distance k from $\mathbf{v}$.

Completely regular and completely transitive codes are classical subjects in algebraic coding theory, which are closely connected with graph theory, combinatorial designs and algebraic combinatorics. Existence, construction and enumeration of all such codes are open hard problems (see [3,6,8,11] and references there).

In this paper we extend our previous construction [13] connecting it with [14] and, as a result, we obtain, for any prime power q, several different infinite classes of completely regular codes with different parameters n, k, q and with identical intersection arrays. This gives different presentations, as coset graphs, of distance-regular bilinear form graphs.

Under the same conditions, an explicit construction of an infinite family of q-ary uniformly packed codes (in the wide sense) with covering radius ρ, which are not completely regular, is also given.

2. Preliminary results

Lemma 2.1 ([11]). *Let C be a linear completely regular code with intersection array $\{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$, and let μ_i be the number of cosets of C of weight i. Then*

$$\mu_{i-1} b_{i-1} = \mu_i c_i.$$

Definition 2.2. For two matrices $A = [a_{r,s}]$ and $B = [b_{i,j}]$ over $\mathbb{F}_q$ define a new matrix H which is the Kronecker product $H = A \otimes B$, where H is obtained by replacing any element $a_{r,s}$ in A by the matrix $a_{r,s}B$.

Consider the matrix $H = A \otimes B$ and let C, C_A and C_B be the codes over $\mathbb{F}_q$ which have, respectively, H, A and B as parity check matrices. Assume that A and B have size $m_a \times n_a$ and $m_b \times n_b$, respectively. Clearly, the codewords in code C are presented as matrices $[\mathbf{c}]$ of size $n_b \times n_a$:

$$[\mathbf{c}] = \begin{bmatrix} c_{1,1} & \ldots & c_{1,n_a} \\ c_{2,1} & \ldots & c_{2,n_a} \\ \vdots & \vdots & \vdots \\ c_{n_b,1} & \ldots & c_{n_b,n_a} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_{n_b} \end{bmatrix} = \left[\mathbf{c}^{(1)}\, \mathbf{c}^{(2)} \ldots \mathbf{c}^{(n_a)}\right], \tag{2}$$

where $c_{i,j} = a_{r,j} b_{s,i}$, $\mathbf{c}_r$ is the rth row vector of the matrix C and $\mathbf{c}^{(\ell)}$ is its ℓth column.

The following result was obtained in [13].