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Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On 2-limited packings of complete grid graphs

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ARTICLE INFO

Article history:
Received 5 November 2015
Received in revised form 15 September 2016
Accepted 1 November 2016
Available online xxxx

Keywords:
Graph packing
2-limited packing
Cartesian product
Box product
Grid graph

ABSTRACT

For a fixed integer t, a set of vertices B of a graph G is a t-limited packing of G provided that the closed neighbourhood of any vertex in G contains at most t elements of B. The size of a largest possible t-limited packing in G is denoted $L_t(G)$ and is the t-limited packing number of G. In this paper, we investigate the 2-limited packing number of Cartesian products of paths. We show that for fixed k the difference $L_2(P_k \square P_n) - L_2(P_k \square P_{n-1})$ is eventually periodic as a function of n, and thereby give closed formulas for $L_2(P_k \square P_n)$, $k = 1, 2, \ldots, 5$. The techniques we use are suitable for establishing other types of packing and domination numbers for Cartesian products of paths and, more generally, for graphs of the form $H \square P_n$.

1. Introduction

Many problems of both practical and theoretical interest involve packing objects into a structure. For example, one might want to place as many radioactive containers as possible into a building without, say, any three containers being too near each other. This might be modeled as the problem of selecting a maximum number of vertices from a graph (location of the containers) such that no graph vertex (location) is adjacent (close) to more than three of the selected vertices (containers).

In [4], the authors introduce the concept of limited packings in a graph, which generalize the well-known concept of closed-neighborhood packings. Recall that a closed-neighborhood packing in a graph G is a subset P of vertices in G with the property that the closed neighborhoods of these vertices are disjoint, or equivalently that the closed neighborhood of each vertex in G contains at most one vertex from the set P. Limited packings generalize this latter point of view.

Definition 1. For a natural number t, a *t-limited packing* B in an undirected graph G is a set of vertices in G such that the closed neighborhood of any vertex in G contains at most t members of B. In other words, B is a t-limited packing in G provided

$$\forall v \in V(G), |N[v] \cap B| \leq t.$$

The problem of selecting graph vertices in the opening paragraph is equivalent to finding a maximum-size 3-limited packing in a graph. In Fig. 1 we illustrate a 2-limited packing in a graph G. The set of shaded vertices B is a 2-limited packing in G.

The graph in Fig. 1 can be obtained as the Cartesian (or box) product of the paths P_3 and P_5; so $G \cong P_3 \square P_5$. Graphs obtained as the Cartesian product of paths are called complete grid graphs, or simply grid graphs.[1] Such graphs naturally arise in applications involving city planning or electrical circuit layouts, for example.

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1 Some authors take 'grid graph' to refer to subgraphs of a complete grid graph; henceforth by grid graph we mean a complete grid graph.

http://dx.doi.org/10.1016/j.disc.2016.11.001

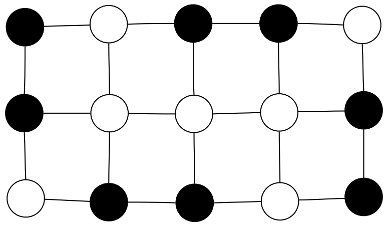

Fig. 1. A 2-limited packing of $P_3\square P_5$.

For a given graph G, the size of a maximum cardinality t-limited packing in G is denoted $L_t(G)$ and is called the *t-limited packing number* of G. The problem of determining whether an arbitrary graph G has a t-limited packing of at least a given size is NP-complete [1]. Limited packings have been discussed by a number of other authors; see for example [2–4,7].

The 2-limited packing shown in Fig. 1 happens to be of largest possible size amongst all 2-limited packings in G, and so $L_2(P_3\square P_5) = 8$. In this paper we are interested in 2-limited packings and, in particular, the value of $L_2(P_k\square P_n)$ for natural numbers n, k. Our motivation for presenting our results for t-limited packings in terms of the case $t = 2$ is that 1-limited packings (i.e. 2-packings) have been well-studied. As a result, it is natural to next consider 2-limiting packings as a concrete example of t-limited packings.

When $k \in \{1, 2\}$ it is easy to determine a formula in n for $L_2(P_k\square P_n)$, and the case $k = 3$ can also be handled relatively simply. These cases will be considered later but, to better understand a more complex situation, consider the case when $k = 4$: Table 1 follows from the results in Section 4.1. This table shows, for small n, the values of $L_2(P_4\square P_n)$ and the values of the growth of this function of n, $\Delta[L_2(P_4\square P_n)] = L_2(P_4\square P_n) - L_2(P_4\square P_{n-1})$.

Table 1
Some 2-limited packing numbers of graphs $P_4\square P_n$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L_2(P_4\square P_n)$	3	4	6	8	10	12	13	15	17	18	20	22	24	25	27
$\Delta[L_2(P_4\square P_n)]$	–	1	2	**2**	**2**	**2**	**1**	**2**	**2**	**1**	2	2	2	1	2

In this paper we show that the difference function $\Delta[L_2(P_4\square P_n)] = L_2(P_4\square P_n) - L_2(P_4\square P_{n-1})$ is eventually periodic as a function of n and that the highlighted '2221221' pattern repeats for $n \geq 4$. From this, a closed formula for $L_2(P_4\square P_n)$ follows easily.

This paper further shows that, for any natural number k, the values $\Delta[L_2(P_k\square P_n)]$ are eventually periodic as a function of n with explicit bounds on when the periodicity begins. As a consequence we give simple formulas (in n) for the 2-limited packing number of $P_k\square P_n$, for each $k \in \{1, 2, 3, 4, 5\}$.

The key idea of the paper is that 2-limited packings in a grid graph can be identified with walks in a certain weighted digraph, so that the size of the packing is the weight of the walk. Thus the computation of the size of maximum 2-limited packings in the grid graph becomes the computation of a maximum-weight walk in a digraph of a given length, which is more straightforward. The intuition that maximum-weight walks of longer and longer lengths must incorporate many "maximum-weight cycles" is developed to show an eventual periodicity in the growth of the size of a maximum 2-limited packing of a grid graph $P_k\square P_n$, as n increases. Bounds are determined on the appearance of the periodicity from which we can determine explicit formulas for the 2-limited packing number of grid graphs $P_k\square P_n$, for $k \in 1, 2, 3, 4, 5$ and any n.

The technique is easily seen to generalize to the computation of 2-limited packing numbers of graphs of the form $G\square P_n$ and, more generally, can be applied to the computation of a number of other graph parameters (such as domination number) on graphs of the form $G\square P_n$. For example, the techniques here show why, for any graph H, the domination number $\gamma(H\square P_n)$ also grows in a regular way (the difference $\Delta[\gamma(H\square P_n)]$ is eventually periodic as a function of n). We note that computation of the domination number of a grid is a non-trivial problem [6].

Similar techniques have been considered by other authors; see for example [5] or [8]. A main contribution of our technique is that we transform the problem to a standard graph problem. Another main contribution of this paper is that we prove the eventual periodicity of the difference function. We have not found another paper which does this. The paper [8] states, on page 7, that a proof of a similar result is to appear in a forthcoming paper but that paper does not seem to ever have been published. Similar work is also discussed in [9], but the methods are quite different than those we use here. Another difference is that we consider the 2-limited packing problem, something not considered in other papers.

2. Packings and maximum-weight walks

The basic idea of this paper is that 2-limited packings in a grid graph are in one-to-one correspondence with weighted directed walks in a particular digraph that we describe in this section. Furthermore, the size of the packing equals the weight of the walk, and so we can reduce the problem of finding a maximum-weight 2-limited packing in a grid graph to that of

Please cite this article in press as: N.E. Clarke, R.P. Gallant, On 2-limited packings of complete grid graphs, Discrete Mathematics (2016), http://dx.doi.org/10.1016/j.disc.2016.11.001