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The expansion of immaculate functions in the ribbon basis

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ABSTRACT

We consider here the algebra of noncommutative symmetric functions, and two of its bases, the ribbon basis and the immaculate basis. Although a simple combinatorial formula is known for expanding an element of the ribbon basis in the immaculate basis, the reverse is not known. Using a sign-reversing involution, we prove an analogue of the classical Jacobi–Trudi formula which is an expression for an immaculate function indexed by a rectangle in the ribbon basis. We generalize this result to immaculate functions indexed by products of rectangles, and use this to prove a combinatorial formula for immaculate functions indexed by a rectangle of the form (2^n).

1. Introduction

In this paper we consider Schur-like bases of the Hopf algebra NSym of noncommutative symmetric functions, and Schur-like elements in the Hopf algebra Sym of symmetric functions. The family of ribbon functions in NSym forms a Schur-like basis of NSym [4]. The skew-Schur functions, and in particular ribbon Schur functions, are closely related to the Schur basis of Sym and properties of this basis involving Ferrers diagrams [14]. Background material concerning the Hopf algebras NSym and Sym is given in [4], and we adopt notation related to these Hopf algebras from [4].

Recently, the *immaculate basis* $\{\mathfrak{S}_\alpha\}_{\alpha\in\mathcal{C}}$ of NSym was introduced in [2], and the *shin basis* $\{\text{ש}_\alpha\}_{\alpha\in\mathcal{C}}$ of NSym was recently introduced in [4]. Both of these bases naturally project onto the Schur basis $\{s_\lambda\}_{\lambda\in\mathcal{P}}$ of Sym. For all $\lambda\in\mathcal{P}$ we have

$$\chi(\mathfrak{S}_\lambda) = \chi(\text{ש}_\lambda) = s_\lambda,$$

where $\chi : \mathsf{NSym} \to \mathsf{Sym}$ is the forgetful projection morphism. That is, $\chi(H_\alpha) = h_{\mathrm{sort}(\alpha)}$ for an element H_α of the complete homogeneous basis of NSym, and $h_{\mathrm{sort}(\alpha)}$ denotes the element of the complete homogeneous basis $\{h_\lambda\}_{\lambda\in\mathcal{P}}$ of Sym indexed by the partition $\mathrm{sort}(\alpha)$.

An elegant combinatorial formula for expanding a ribbon function in the immaculate basis is proven in [2]. The ribbon basis $\{R_\alpha\}_{\alpha\in\mathcal{C}}$ has a positive expansion in the immaculate basis, with

$$R_\beta = \sum_{\alpha\geq_\ell\beta} L_{\alpha,\beta}\mathfrak{S}_\alpha,$$

where $\leq_\ell$ denotes the lexicographic order on $\mathcal{C}$ and $L_{\alpha,\beta}$ denotes the number of standard immaculate tableaux of shape α and descent composition β. However, there is no known combinatorial formula of the form

$$\mathfrak{S}_\beta = \sum_{\alpha\geq_\ell\beta} M_{\alpha,\beta}R_\alpha$$

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for expanding immaculate functions in the ribbon basis, and it is natural to consider combinatorial properties associated with coefficients of the form $M_{\alpha,\beta}$.

We currently leave it as an open problem to construct a general combinatorial formula for coefficients of the form $M_{\alpha,\beta}$, and we leave it as an open problem to prove a general combinatorial formula for expanding shin functions in the ribbon basis. In this paper we prove several formulas for expanding certain classes of immaculate functions in the ribbon basis. In particular, we use a simple sign-reversing involution to prove the following theorem.

Theorem 1. *The identity*

$$\mathfrak{S}_{(m^n)} = \sum_{\sigma \in S_n} (-1)^{\sigma} R_{(m-1+\sigma_1, m-2+\sigma_2, \ldots, m-n+\sigma_n)}$$

holds for a rectangle $(m^n) \in \mathcal{C}$.

Although our formulas for immaculate functions are of interest in their own right, there is an obvious application of such formulas in terms of commutative Schur functions. In particular, we obtain new formulas for expressing commutative Schur functions such as Schur-rectangles as linear combinations of ribbon Schur functions. Applying the projection morphism χ to both sides of Theorem 1, we obtain the following Jacobi–Trudi-like formula for commutative Schur-rectangles:

$$s_{(m^n)} = \sum_{\sigma \in S_n} (-1)^{\sigma} r_{(m-1+\sigma_1, m-2+\sigma_2, \ldots, m-n+\sigma_n)}.$$

Lascoux and Pragacz in 1988 proved a formula for a Schur function in terms of a determinant consisting of ribbon Schur functions, and in 1995 Hamel and Goulden proved a much more general formula for a Schur function in terms of a determinant consisting of ribbon Schur functions [7,14]. Determining the coefficients arising in the expansion of a Schur function as a linear combination of ribbon Schur functions is generally nontrivial. Of course, determinantal expressions given by the Lascoux–Pragacz identity and the Hamel–Goulden identity may be expanded as a linear combination of ribbon Schur functions using the classical *ribbon Schur multiplication rule*. This is the product rule whereby

$$r_{\alpha} r_{\beta} = r_{\alpha \cdot \beta} + r_{\alpha \odot \beta},$$

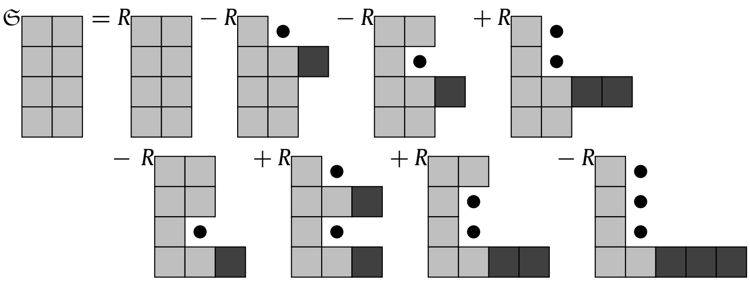
where $\alpha \cdot \beta$ denotes the concatenation of α and β and $\alpha \odot \beta$ denotes the near concatenation of α and β. However, determining the coefficients resulting from such expansions is difficult, especially for larger determinants.

In Section 3, we generalize Theorem 1 for immaculate functions indexed by certain products of rectangles. Observe that it is not in general true that

$$\mathfrak{S}_{\alpha} = \sum_{\sigma \in S_{\ell(\alpha)}} (-1)^{\sigma} R_{(\alpha_1 - 1 + \sigma_1, \alpha_2 - 2 + \sigma_2, \ldots, \alpha_{\ell(\alpha)} - \ell(\alpha) + \sigma_{\ell(\alpha)})}$$

for $\alpha \in \mathcal{C}$. We presently leave the classification of compositions α such that the above equality holds as an open problem.

In Section 4, we prove a combinatorial rule for writing an immaculate function of the form $\mathfrak{S}_{(2^n)}$ as a linear combination of noncommutative ribbon Schur functions using our main result. This combinatorial rule shows how an immaculate-rectangle of the form $\mathfrak{S}_{(2^n)}$ may be written as a sum of 2^{n-1} elements in $\{R_{\alpha}\}_{\alpha \in \mathcal{C}}$ indexed by compositions defined in terms of a partial order $\trianglelefteq$ we construct on $\mathcal{C}$. In contrast, our general formula for immaculate-rectangles is a sum over permutations of n and hence has as many as $n!$ terms. Our combinatorial formula for expressions of the form $\mathfrak{S}_{(2^n)}$ is illustrated below. We henceforward use French notation for diagrams, whereby the first entry of a composition is at the bottom of the diagram of the composition.



Write $\alpha \blacktriangleleft \beta$ if diag(β) can be obtained from diag(α) by moving the rightmost cell of a row in diag(α) consisting of two cells in a diagonal "strictly southeast" direction to the nearest row consisting of at least two cells, as illustrated in the example

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