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1-perfectly orientable graphs and graph products

Tatiana Romina Hartinger, Martin Milanič *

University of Primorska, UP IAM, Muzejski trg 2, SI-6000 Koper, Slovenia
University of Primorska, UP FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia

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ABSTRACT

A graph G is said to be *1-perfectly orientable* (1-p.o. for short) if it admits an orientation such that the out-neighborhood of every vertex is a clique in G. The class of 1-p.o. graphs forms a common generalization of the classes of chordal and circular arc graphs. Even though 1-p.o. graphs can be recognized in polynomial time, no structural characterization of 1-p.o. graphs is known. In this paper we consider the four standard graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product. For each of them, we characterize when a nontrivial product of two graphs is 1-p.o.

1. Introduction

A *tournament* is an orientation of a complete graph. We study graphs having an orientation that is an *out-tournament*, that is, a digraph in which the out-neighborhood of every vertex induces a *tournament*. (*In-tournaments* are defined similarly.) Following the terminology of Kammer and Tholey [11], we say that an orientation of a graph is 1-*perfect* if it is an out-tournament, and that a graph is 1-*perfectly orientable* (1-*p.o.* for short) if it has a 1-perfect orientation. In [11], Kammer and Tholey introduced the more general concept of *k-perfectly orientable* graphs, as graphs admitting an orientation in which the out-neighborhood of each vertex can be partitioned into at most k sets each inducing a tournament. They developed several approximation algorithms for optimization problems on k-perfectly orientable graphs and related classes. It is easy to see (simply by reversing the arcs) that 1-p.o. graphs are exactly the graphs that admit an orientation that is an in-tournament. In-tournament orientations were called *fraternal orientations* in several papers [3–6,12,13,16].

The concept of 1-p.o. graphs was introduced in 1982 by Skrien [15] (under the name $\{B_2\}$-*graphs*), where the problem of characterizing 1-p.o. graphs was posed. While a structural understanding of 1-p.o. graphs is still an open question, partial results are known. Bang-Jensen et al. observed in [1] that 1-p.o. graphs can be recognized in polynomial time via a reduction to 2-SAT. Skrien [15] characterized graphs admitting an orientation that is both an in-tournament and an out-tournament as exactly the proper circular arc graphs. All chordal graphs and all circular arc graphs are 1-p.o. [16], and, more generally, so is any vertex-intersection graph of connected induced subgraphs of a unicyclic graph [1,14]. Every graph having a unique induced cycle of order at least 4 is 1-p.o. [1].

In [8], several operations preserving the class of 1-p.o. graphs were described (see Section 2); operations that do not preserve the property in general were also considered. In the same paper 1-p.o. graphs were characterized in terms of

* Corresponding author at: University of Primorska, UP FAMNIT, Glagoljaška 8, SI-6000 Koper, Slovenia.
E-mail addresses: tatiana.hartinger@iam.upr.si (T.R. Hartinger), martin.milanic@upr.si (M. Milanič).

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edge-clique covers, and characterizations of 1-p.o. cographs and of 1-p.o. co-bipartite graphs were given. In particular, a cograph is 1-p.o. if and only if it is $K_{2,3}$-free and a co-bipartite graph is 1-p.o. if and only if it is circular arc. A structural characterization of line graphs that are 1-p.o. was given in [1].

In this paper we consider the four standard graph products: the Cartesian product, the strong product, the direct product, and the lexicographic product. For each of these four products, we completely characterize when a nontrivial product of two graphs G and H is 1-p.o. While the results for the Cartesian, the lexicographic, and the direct products turn out to be rather straightforward, the characterization for the case of the strong product is more involved.

Some common features of the structure of the factors involved in the characterizations can be described as follows. In the cases of the Cartesian and the direct product the factors turn out to be very sparse and very restricted, always having components with at most one cycle. In the case of the lexicographic and of the strong product the factors can be dense. More specifically, co-bipartite 1-p.o. graphs, including co-chain graphs in the case of strong products, play an important role in these characterizations. The case of the strong product also leads to a new infinite family of 1-p.o. graphs (cf. Proposition 6.8).

The paper is organized as follows. Section 2 includes the basic definitions and notation, and recalls several known results about 1-p.o. graphs that will be required for some of the proofs. In Sections 3–6 we deal, respectively, with 1-p.o. Cartesian product graphs, 1-p.o. lexicographic product graphs, 1-p.o. direct product graphs, and 1-p.o. strong product graphs, and state and prove the corresponding characterizations.

2. Preliminaries

All graphs considered in this paper are simple and finite, but may be either undirected or directed (in which case we refer to them as digraphs). An edge in a graph connecting vertices u and v will be denoted simply uv. The *neighborhood* of a vertex v in a graph G is the set of all vertices adjacent to v and will be denoted by $N_G(v)$. The *degree* of v is the size of its neighborhood. A *leaf* in a graph is a vertex of degree 1. The *closed neighborhood* of v in G is the set $N_G(v) \cup \{v\}$, denoted by $N_G[v]$. An *orientation* of a graph $G = (V, E)$ is a digraph $D = (V, A)$ obtained by assigning a direction to each edge of G. Given a digraph $D = (V, A)$, the *in-neighborhood* of a vertex v in D, denoted by $N_D^-(v)$, is the set of all vertices w such that $(w, v) \in A$. Similarly, the *out-neighborhood* of v in D is the set $N_D^+(v)$ of all vertices w such that $(v, w) \in A$. We may omit the subscripts when the corresponding graph or digraph is clear from the context. Given an undirected graph G and a set $S \subseteq V(G)$, we define the *neighborhood of S* as $N(S) = (\bigcup_{x\in S} N(x)) \setminus S$. The *subgraph of G induced by S* is the graph, denoted by $G[S]$, with vertex set S and edge set $\{uv : u \in S, v \in S, uv \in E(G)\}$. The *distance* between two vertices x and y in a connected graph G will be denoted by $d_G(x, y)$ (or simply $d(x, y)$) and defined, as usual, as the length of a shortest x–y path.

Given two graphs G and H, their *union* is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Their *disjoint union* is the graph $G + H$ with vertex set $V(G) \dot\cup V(H)$ (disjoint union) and edge set $E(G) \cup E(H)$ (if G and H are not vertex disjoint, we first replace one of them with a disjoint isomorphic copy). We write $2G$ for $G + G$. The *join* of two graphs G and H is the graph denoted by $G * H$ and obtained from the disjoint union of G and H by adding to it all edges joining a vertex of G with a vertex of H. Given two graphs G and H and a vertex v of G, the *substitution* of v in G for H consists in replacing v with H and making each vertex of H adjacent to every vertex in $N_G(v)$ in the new graph.

A *clique* (resp., *independent set*) in a graph G is a set of pairwise adjacent (resp., non-adjacent) vertices of G. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G in which two distinct vertices are adjacent if and only if they are not adjacent in G. The fact that two graphs G and H are isomorphic to each other will be denoted by $G \cong H$. Given a family $\mathcal{F}$ of graphs, we say that a graph is *$\mathcal{F}$-free* if it has no induced subgraph isomorphic to a graph of $\mathcal{F}$.

K_n, C_n and P_n denote the n-vertex complete graph, cycle, and path, respectively. The *claw* is the complete bipartite graph $K_{1,3}$, that is, a star with 3 edges, 3 leaves and one central vertex. The *bull* is a graph with 5 vertices and 5 edges, consisting of a triangle with two disjoint pendant edges. The *gem* is the graph $P_4 * K_1$, that is, the 5-vertex graph consisting of a 4-vertex path plus a vertex adjacent to each vertex of the path.

For graph theoretic notions not defined above, see, e.g. [17]. We will recall the definitions and some basic facts about each of the four graph products studied in the respective sections (Sections 3–6). For each of the four considered products, we say that the product of two graphs is *nontrivial* if both factors have at least 2 vertices. For further details regarding product graphs and their properties, we refer to [7,10].

In [8], several results about 1-p.o. graphs were proved. In the rest of this section we list some of them for later use.

Proposition 2.1. *No graph in the set $\{F_1, F_2, F_3, F_4\}$ (see Fig. 1) is 1-perfectly orientable.*

Two distinct vertices u and v in a graph G are said to be *true twins* if $N_G[u] = N_G[v]$. We say that a vertex v in a graph G is *simplicial* if its neighborhood forms a clique and *universal* if it is adjacent to all other vertices of the graph, that is, if $N_G[v] = V(G)$. The operations of adding a true twin, a universal vertex, or a simplicial vertex to a given graph are defined in the obvious way.

Proposition 2.2. *The class of 1-p.o. graphs is closed under each of the following operations:*

(a) *Disjoint union.*
(b) *Adding a true twin.*
(c) *Adding a universal vertex.*
(d) *Adding a simplicial vertex.*

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