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Monochromatic connecting colorings in strongly connected oriented graphs

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ABSTRACT

An arc-coloring of a strongly connected digraph D is a *strongly monochromatic-connecting coloring* if for every pair u, v of vertices in D there exist an (u, v)-monochromatic path and a (v, u)-monochromatic path. Let $smc(D)$ denote the maximum number of colors that can be used in a strongly monochromatic-connecting coloring of D. In this paper we prove that if D is a strongly connected oriented graph of size m, then $smc(D) = m - \Omega(D) + 1$, where $\Omega(D)$ is the minimum size of a spanning strongly connected subdigraph of D. As a corollary of this result, we see that a strongly connected oriented graph D of order n is hamiltonian if and only if $smc(D) = m - n + 1$.

1. Introduction

An interesting generalization of the concept of connectivity in graphs, due to Chartrand, Johns, McKeon and Zhang [5], is rainbow connecting colorings. An edge-colored graph G is *rainbow connected* if there exists a path, with no two edges colored the same, between any two vertices of G (see for instance [7]), and this concept can be easily extended to digraphs (see [1,6]). The *rainbow connection number* of a graph G is the minimum number of colors that are needed in order to make G rainbow connected. The rainbow connection number, besides being an interesting combinatorial measure, has applications to the secure transfer of classified information between agencies and in the area of networking. This applications background is from Li and Sun [7]. Caro and Yuster [4], as a naturally related question, introduced the concept of monochromatic-connecting coloring of a graph. An edge-coloring of a graph G is a *monochromatic-connecting coloring* if there exists a monochromatic path between any two vertices of G. The above definition can be also naturally extended to digraphs. An arc-coloring of a digraph D is a *strongly monochromatic-connecting coloring* (SMC coloring, for short) if for every pair u, v of vertices in D there exist an (u, v)-monochromatic path and a (v, u)-monochromatic path. The *strong monochromatic connection number* of a strongly connected digraph D, denoted by $smc(D)$, is defined as the maximum number of colors used in an SMC coloring of D.

Observe that given a strongly connected digraph D and a strongly connected spanning subdigraph H of D, by coloring the arcs of H with one single color and the remaining arcs with distinct colors, we obtain an SMC coloring of D with $m-|A(H)|+1$ colors, and therefore $smc(D) \geq m - |A(H)| + 1$. In this paper the following theorem is proved.

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Theorem 1. *Let D be a strongly connected oriented graph of size m, and let $\Omega(D)$ be the minimum size of a strongly connected spanning subdigraph of D. Then*

$$smc(D) = m - \Omega(D) + 1.$$

As a corollary of Theorem 1, it follows that an oriented graph D of order n is hamiltonian if and only if $smc(D) = m-n+1$. From here we can see also that computing $\Omega(D)$ is NP-hard.

2. Definitions and notation

All the digraphs considered in this work are finite oriented graphs, that is, a digraph with no symmetric arcs or loops. A digraph is said to be *connected* if its underlying graph is connected. Given a digraph $D = (V(D), A(D))$, a vertex v is *reachable* from a vertex u if D contains an (u, v)-path. A digraph D is *unilateral* if, for every pair u, v of vertices of D, either u is reachable from v or v is reachable from u (or both). A digraph D is *strongly connected* if for every pair of vertices $\{u, v\} \subseteq V(D)$, the vertex u is reachable from v and the vertex v is reachable from u. Given a strongly connected digraph D, let $\Omega(D)$ denote the minimum size of a strongly connected spanning subdigraph of D. The *minimum in-degree* (resp. *out-degree*) of a subdigraph H of D will be denoted as $\delta^-(H)$ (resp. $\delta^+(H)$).

Let $D = (V(D), A(D))$ be a strongly connected digraph. Given $S \subseteq V(D)$, the *subdigraph induced* by S is the subdigraph of D of maximum size which has S as set of vertices, and will be denoted as $D[S]$. Given $F \subseteq A(D)$, the *subdigraph induced* by F is the subdigraph of D of minimum order which has F as set of arcs, and will be denoted as $D[F]$. Given a positive integer p, let $[p] = \{1, 2, \dots, p\}$. As a *p-coloring* of D we will understand a surjective function $\Gamma : A(D) \to [p]$. For each "color" $i \in [p]$ the set of arcs $\Gamma^{-1}(i)$ will be called the *chromatic class* (of color i), and if $|\Gamma^{-1}(i)| = 1$, the color i and the chromatic class $\Gamma^{-1}(i)$ will be called *trivial*. For each Γ, k_Γ will be the number of non-trivial colors of Γ and, if $k_\Gamma \geq 1$, we will assume that $[k_\Gamma] = \{1, \dots, k_\Gamma\} \subseteq [p]$ is the set of non-trivial colors.

A subdigraph H of D will be called *monochromatic* if $A(H)$ is contained in a chromatic class, and we will say that the color i appears in H if $A(H) \cap \Gamma^{-1}(i) \neq \emptyset$.

A p-coloring of a digraph D is a *strongly monochromatic-connecting coloring* (SMC coloring, for short) if for every pair u, v of vertices in D there exist an (u, v)-monochromatic path and a (v, u)-monochromatic path.

The *strong monochromatic connection number* of a strongly connected digraph D, denoted by $smc(D)$, is defined as the maximum integer p such that there is a p-coloring of D which is an SMC coloring of D.

We will say that a p-coloring Γ of D is a *good coloring* if $p = smc(D)$ and Γ is an SMC coloring of D.

For general concepts we may refer the reader to [2,3].

3. The results

In this section we present the proof of Theorem 1. We start with a remark and a lemma that will be used in the proof of Theorem 1.

Remark 2. Let $D = (V(D), A(D))$ be a strongly connected oriented graph and let Γ be a SMC coloring of D.

1. Since D has no symmetric arcs, for each $x \in V(D)$ there is at least one arc colored with a non-trivial color which is incident to x.
2. If Γ is a good coloring of D, for every non-trivial color i, $D[\Gamma^{-1}(i)]$ is connected.
3. Let $F \subseteq A(D)$ such that $D[F]$ is strongly connected and let $I = \{\Gamma(e) : e \in F\}$. Let $V = V(D[F])$ and $F' = A(D[V]) \setminus F$. Given $i \in I$, consider the coloring Γ' such that: $\Gamma'(e) = \Gamma(e)$ for every $e \in A(D)$ such that $\Gamma(e) \notin I$; $\Gamma'(e) = i$ for every $e \in A(D) \setminus F'$ such that $\Gamma(e) \in I$; and each $e \in F'$ such that $\Gamma(e) \in I$ receive a trivial color. Γ' is a SMC coloring.

Lemma 3. *Let $D = (V(D), A(D))$ be a strongly connected oriented graph of order n. Then there is a good coloring Γ of D such that each non-trivial chromatic class of Γ induces a strongly connected subdigraph of D.*

Moreover, if Γ have at least 2 non-trivial chromatic classes, then each non-trivial chromatic class induces a subdigraph of order q, with $3 \leq q \leq n-2$.

Proof. Let $D = (V(D), A(D))$ be a strongly connected oriented graph of order n, and let t_D be the maximum number of trivial colors that appear in a good coloring of D.

Given a good coloring Γ of D, for each non-trivial color $i \in [k_\Gamma]$ let $r_\Gamma(i)$ be the number of ordered pairs of vertices (x, y) of the subdigraph $D[\Gamma^{-1}(i)]$ such that there is no (x, y)-path in $D[\Gamma^{-1}(i)]$, and let $r_\Gamma = \sum_{i\in[k_\Gamma]} r_\Gamma(i)$.

Let Γ be a good coloring of D with t_D trivial colors and such that, among all the good colorings of D with t_D trivial colors, r_Γ is minimum.

For each $i \in [k_\Gamma]$, let $H_i = D[\Gamma^{-1}(i)]$ and let $D^* = D[\bigcup_{i\in[k_\Gamma]} \Gamma^{-1}(i)]$.

Claim 1. If $uv \in A(D^*)$ and P is an (u, v)-trail in $D^* - \{uv\}$, then at least 3 colors appear in the subgraph $P \cup \{uv\}$.

Let $uv \in A(D^*)$ and P be an (u, v)-trail in $D^* - \{uv\}$. Since $uv \in A(D^*)$ it follows there is $i \in [k_\Gamma]$ such that $uv \in A(H_i)$. If P is an (u, v)-trail in $H_i - \{uv\}$, the coloring Γ' obtained from Γ by assigning a new color to the arc uv is an SMC coloring of