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Local connectivity, local degree conditions, some forbidden induced subgraphs, and cycle extendability

Christoph Brause [a], Dieter Rautenbach [b,*], Ingo Schiermeyer [a]

[a] Institute of Discrete Mathematics and Algebra, TU Bergakademie Freiberg, Freiberg, Germany
[b] Institute of Optimization and Operations Research, Ulm University, Ulm, Germany

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ABSTRACT

The research in the present paper was motivated by the conjecture of Ryjáček that every locally connected graph is weakly pancyclic. We show that the conjecture holds for several classes of graphs.

In particular, for a connected, locally connected graph G of order at least 3, our results are as follows: If G is $(K_1+(K_1\cup K_2))$-free, then G is weakly pancyclic. If G is $(K_1+(K_1\cup K_2))$-free, then G is fully cycle extendable if and only if $2\delta(G)\geq n(G)$. If G is $\{K_1+K_1+\bar{K}_3, K_1+P_4\}$-free or $\{K_1+K_1+\bar{K}_3, K_1+(K_1\cup P_3)\}$-free, then G is fully cycle extendable. If G is distinct from $K_1+K_1+\bar{K}_3$ and $\{K_1+P_4, K_{1,4}, K_2+(K_1\cup K_2)\}$-free, then G is fully cycle extendable. Furthermore, we prove that a degree condition weaker than locally Dirac or locally Ore guarantees fully cycle extendability.

1. Introduction

The research in the present paper was motivated by the conjecture of Ryjáček that every locally connected graph is weakly pancyclic. Before showing that the conjecture holds for several classes of graphs, we collect some notation and standard terminology.

We consider finite, simple, and undirected graphs. A graph G is *locally connected* if for every vertex u of G, the subgraph $G[N_G(u)]$ of G induced by the neighborhood $N_G(u)$ of u in G is connected. Similarly, G is *locally Ore* if $|N_G(u)\cap N_G(v)|+|N_G(u)\cap N_G(w)|\geq d_G(u)$ for every induced path vuw of order 3 in G, and G is *locally Dirac* if $2\delta(G[N_G(u)])\geq d_G(u)$ for every vertex u of G, where $\delta(G)$ and $d_G(u)$ are the minimum degree of G and the degree of u in G, respectively. Furthermore, a graph G is locally isometric if $G[N_G(v)]$ is of diameter at most 2 for every vertex $v\in V(G)$. A graph G is *k-connected* if $G-S$ is connected for every set of vertices S with cardinality at most $k-1$.

For a vertex u of a graph G and some positive integer k, let $N_G^k(u)$ be the set of vertices of G at distance exactly k from u. For a graph G, the *girth* $g(G)$ and the *circumference* $c(G)$ are the minimum and the maximum order of a cycle in G, respectively. A graph G is *hamiltonian* if $c(G)=n(G)$, where $n(G)$ is the order of G. A graph G is *weakly pancyclic* if it has a cycle of order ℓ for every integer ℓ between $g(G)$ and $c(G)$. A cycle C in a graph G is *extendable* if G contains a C' of order $n(C)+1$ with $V(C)\subseteq V(C')$. A graph G is *fully cycle extendable* if every vertex of G lies on a triangle, and every cycle in G of order less than $n(G)$ is extendable. For two disjoint graphs G and H, let $G\cup H$ be their union, let $G+H$ be their join, that is, the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{uv : u\in V(G), v\in V(H)\}$. Let $\bar{G}$ be the complement of G. Let K_n, P_n, and C_n be the complete graph, path, and cycle of order n, respectively. Note that the star $K_{1,n-1}$ of order n is $K_1+\bar{K}_{n-1}$. Let

* Corresponding author.
E-mail addresses: brause@math.tu-freiberg.de (C. Brause), dieter.rautenbach@uni-ulm.de (D. Rautenbach), schiermeyer@math.tu-freiberg.de (I. Schiermeyer).

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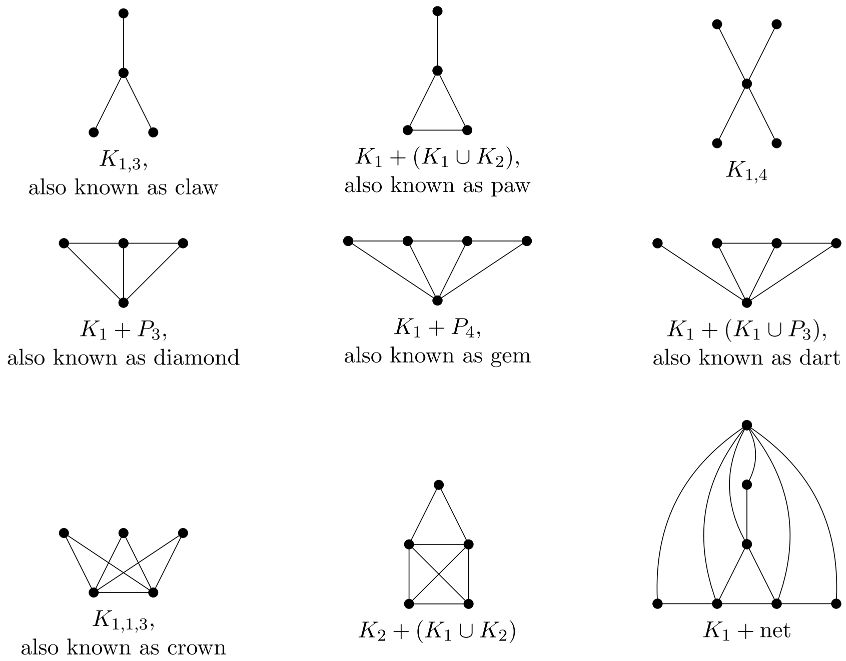

Fig. 1. Forbidden induced subgraphs.

$K_{1,1,n-2} = K_1 + K_1 + \bar{K}_{n-2}$. Note that Fig. 1 contains a set of forbidden induced subgraphs used in our results. If $\mathcal{F}$ is a set of graphs, then a graph is *$\mathcal{F}$-free* if it does not contain a graph from $\mathcal{F}$ as an induced subgraph. If $\mathcal{F}$ contains only one graph F, we write *F-free* instead of $\mathcal{F}$-free.

When people started to study sufficient and necessary conditions for graphs being hamiltonian or pancyclic, forbidden induced subgraphs and degree conditions were considered. For example, Bedrossian [3] gave a full characterization of all pairs of connected forbidden induced subgraphs which imply hamiltonicity, and all pairs of connected forbidden induced subgraphs which imply pancyclicity in the class of 2-connected graphs. Furthermore, the degree-conditions of Ore [12] and Dirac [6] are central in the field of hamiltonian graphs.

Note that any connected, locally connected graph is 2-connected, every fully cycle extendable graph is pancyclic, and every pancyclic graph is weakly pancyclic. Motivated by Bedrossian's characterization and these relations, we characterize the graphs F with the property that any connected, locally connected, F-free graph is fully cycle extendable. Furthermore, we identify sets $\mathcal{F}$ of graphs such that any connected, locally connected, $\mathcal{F}$-free graph is fully cycle extendable. In addition, our results show that the following conjecture of Ryjáček holds for several classes of graphs.

Conjecture 1 (*Ryjáček [13]*)**.** *Every locally connected graph is weakly pancyclic.*

We show further that Conjecture 1 holds for graphs that fulfill weaker conditions than being locally Ore or locally Dirac.

Before we proceed to our results, we give a very short summary of previous related work. Chartrand and Pippert [4] showed that every connected, locally connected graph G with $n(G) \geq 3$ and maximum degree $\Delta(G) \leq 4$ is either hamiltonian or $K_{1,1,3}$. Extending a result of Kikust [10], Hendry [9] showed that every connected, locally connected graph G with $n(G) \geq 3$, $\Delta(G) \leq 5$, and $\Delta(G) - \delta(G) \leq 1$ is fully cycle extendable. Gordon, Orlovich, Potts, and Strusevich [8] generalized this last result further to graphs G with $n(G) \geq 3$, $\Delta(G) \leq 5$, and $\delta(G) \geq 3$. Extending earlier results due to Oberly, Sumner [11], and Clark [5], Zhang [14] showed that every connected, locally connected, $K_{1,3}$-free graph G with $n(G) \geq 3$ is fully cycle extendable. Faudree, Ryjáček, and Schiermeyer [7] weakened the local connectivity requirement for this last result. Asratian [1] considered graphs with $|N_G(u) \cap N_G(v) \cap N_G(w)| > |N_G(u) \setminus (N_G[u] \cup N_G[w])|$ for every induced path uvw of order 3, and showed that these graphs are fully cycle extendable. Our results generalize the mentioned results of Zhang [14] and Asratian [1,2].

2. Forbidden induced subgraph conditions

Our first goal is to prove Conjecture 1 for $(K_1 + (K_1 \cup K_2))$-free graphs. Surprisingly, it turns out that, depending on the minimum degree of a connected, locally connected, $(K_1 + (K_1 \cup K_2))$-free graph G, the graph G is either fully cycle extendable or non-hamiltonian.