Discrete Mathematics 340 (2017) 637–643

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Two-geodesic-transitive graphs which are locally connected[☆]

Wei Jin

School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China
Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China

ARTICLE INFO

Article history:
Received 25 April 2016
Received in revised form 6 October 2016
Accepted 30 November 2016
Available online 3 January 2017

Keywords:
Two-geodesic-transitive graph
Quasiprimitive permutation group
Automorphism group

ABSTRACT

We investigate the family of 2-geodesic-transitive graphs which are locally connected. Let Γ be such a graph. It is first shown that: for any integer $d \geq 2$, there exists such a Γ of diameter d; for any integer $k \geq 3$, there exists such a Γ of valency k unless k is a prime and $k \equiv 3 \pmod 4$. Next, we completely determine the family of 2-geodesic-transitive graphs which are locally isomorphic to $\overline{mC_n}$ for some $m \geq 1, n \geq 3$. Finally, we give a reduction result for the family of locally connected $(G, 2)$-geodesic-transitive graphs.

1. Introduction

In this paper, all graphs are finite, simple, connected and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *automorphism group*, respectively. For the group theoretic terminology not defined here we refer the reader to [3,9,25]. An *arc* is an ordered pair of adjacent vertices. A regular graph Γ is said to be *arc-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the set of arcs. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*geodesic* if $u \neq w$ and u, w are not adjacent. An arc-transitive graph Γ is said to be 2-*geodesic-transitive* if it has 2-geodesics, and $\mathrm{Aut}(\Gamma)$ is transitive on the set of 2-geodesics. The possible local structures of 2-geodesic-transitive graphs are characterized in [7], and the families of 2-geodesic-transitive graphs of valency 4 and of prime valency have been determined in [6] and [8], respectively. The graph in Fig. 1 is the octahedron which is the smallest locally connected 2-geodesic-transitive graph.

The *diameter* of a connected graph Γ is the maximal distance between any pair of vertices, and denoted by $\mathrm{diam}(\Gamma)$. A subgraph X of a graph Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U. For a vertex u of Γ, we denote by $\Gamma_i(u)$ the set of vertices at distance i from u in Γ and we set $\Gamma(u) = \Gamma_1(u)$. Let Σ be a graph. For a positive integer m, the graph consisting of m vertex disjoint copies of Σ is denoted by $m\Sigma$. Devillers, Li, Praeger and the author [7, Theorem 1.1] proved that if Γ is a 2-geodesic-transitive graph of valency at least 2, then for a vertex u, either

(1) $[\Gamma(u)] \cong m\mathrm{K}_r$ for some integers $m \geq 2, r \geq 1$; or

(2) $[\Gamma(u)]$ is a connected graph of diameter 2.

Further, Theorem 1.4 of [7] shows that there is a bijection between the family of locally disconnected 2-geodesic transitive graphs Γ and a certain family of partial linear spaces $\mathcal{S}(\Gamma)$. In this paper, we study the family of 2-geodesic-transitive graphs which are locally connected. Thus, for every vertex u, the induced subgraph $[\Gamma(u)]$ is a connected graph of diameter 2.

Remark 1.1. A vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$. A graph Γ is said to be 2-*arc-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on both arcs and 2-arcs. The family of 2-arc-transitive graphs

☆ Supported by the NSF of China (11561027, 11661039, 61563018) and NSF of Jiangxi (GJJ150460, GJJ150444, 20151BAB201001).
E-mail address: jinwei@jxufe.edu.cn.

http://dx.doi.org/10.1016/j.disc.2016.11.036

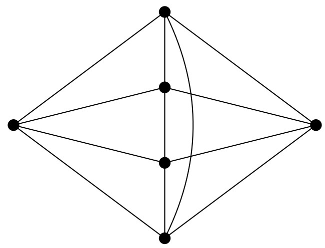

Fig. 1. Octahedron.

has been studied intensively, beginning with the seminal result of Tutte [21,22], for more work see [10–12,16,20,23,24]. By definition, each 2-geodesic of Γ is a 2-arc, but the converse is not true, for instance 2-arcs in a triangle of Γ are not 2-geodesics. Thus the family of non-complete 2-arc-transitive graphs is properly contained in the family of 2-geodesic-transitive graphs. We remark that every locally connected 2-geodesic-transitive graph has girth 3, so it is not 2-arc-transitive.

If k is a prime such that $k \equiv 3 \pmod 4$, then there is no 2-geodesic-transitive graph of valency k, see [8]. Note that a 2-geodesic-transitive graph has diameter at least 2, and the cycle C_n with $n \geq 4$ is not locally connected. We have the following result.

Theorem 1.2. (1) *For any integer $d \geq 2$, there exists a locally connected* 2*-geodesic-transitive graph of diameter d.*

(2) *For any integer $k \geq 3$, there exists a locally connected* 2*-geodesic-transitive graph of valency k unless k is a prime and* $k \equiv 3 \pmod 4$*.*

The *complement graph* $\overline{\Sigma}$ of a graph Σ, is the graph with vertex $V(\Sigma)$, and two vertices are adjacent in $\overline{\Sigma}$ if and only if they are not adjacent in Σ. The families of locally connected 2-geodesic-transitive graphs having prime valency and twice a prime valency were classified in [8] and [13], respectively. Our second theorem determines another special family of locally connected 2-geodesic-transitive graphs, that is, the subgraph induced by the neighbourhood of a vertex is isomorphic to $\overline{mC_n}$ for some $m \geq 1, n \geq 3$.

Let $\Omega = \{1, \ldots, n\}$, where $n \geq 3$, and let $1 \leq k \leq [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then the *Johnson graph* $J(n, k)$ is the graph whose vertex set is the set of all k-subsets of Ω, and two vertices u and v are adjacent if and only if $|u \cap v| = k - 1$. For two integers $m \geq 3$, $b \geq 2$, let $\mathrm{K}_{m[b]}$ denote the *complete multipartite graph*, whose vertex set consisting of m parts of size b, with edges between all pairs of vertices from distinct parts. A graph Γ is said to be *locally isomorphic* to a graph Σ if, for every $u \in V(\Gamma)$, the induced subgraph $[\Gamma(u)]$ is isomorphic to Σ.

Theorem 1.3. *Let Γ be a connected* 2*-geodesic-transitive graph which is locally isomorphic to $\overline{mC_n}$ for some $m \geq 1, n \geq 3$. Then Γ is one of $J(5, 2)$, $\mathrm{K}_{(m+1)[3]}$ or the icosahedron.*

Let G be a group of permutations acting on $\Omega = V(\Gamma)$. Let N be an intransitive normal subgroup of G and let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ be the set of N-orbits in Ω. Then the *normal quotient* Γ_N of Γ is the graph with vertex set $\mathcal{B}$ such that $\{B_i, B_j\}$ is an edge of Γ_N if and only if there exist $x \in B_i$, $y \in B_j$ such that $\{x, y\}$ is an edge of Γ. The graph Γ is said to be a *cover* of Γ_N if, for each edge $\{B_i, B_j\}$ of Γ_N and $v \in B_i$, we have $|\Gamma(v) \cap B_j| = 1$.

A transitive permutation group G is said to be *quasiprimitive*, if every non-trivial normal subgroup of G is transitive. For knowledge of quasiprimitive permutation groups refer to [20] and [19]. Praeger [20] divided the family of quasiprimitive permutation groups into 8 distinct types: Holomorph Affine (HA), Almost Simple (AS), Twisted Wreath product (TW), Product Action (PA), Simple Diagonal (SD), Holomorph Simple (HS), Holomorph Compound (HC) and Compound Diagonal (CD).

Our third theorem is a reduction result on the family of locally connected 2-geodesic-transitive graphs. A graph Γ is said to be $(G, 2)$-*geodesic-transitive* for some $G \leq \mathrm{Aut}(\Gamma)$, if Γ has at least one 2-geodesic, and G acts transitively on the vertex set, the set of arcs and the set of 2-geodesics.

Theorem 1.4. *Let Γ be a connected $(G, 2)$-geodesic-transitive graph which is also locally connected. Then one of the following holds.*

(1) *Γ is isomorphic to $\mathrm{K}_{m[b]}$ for some $m \geq 3, b \geq 2$.*

(2) *G is quasiprimitive on $V(\Gamma)$.*

(3) *G is not quasiprimitive on $V(\Gamma)$, and G has an intransitive normal subgroup N such that: Γ is a cover of Γ_N, G/N is quasiprimitive on $V(\Gamma_N)$, and either Γ_N is complete G/N-arc-transitive or Γ_N is $(G/N, 2)$-geodesic-transitive and locally connected.*