Discrete Mathematics 340 (2017) 678–690

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Degree sum conditions for vertex-disjoint cycles passing through specified vertices

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ARTICLE INFO

Article history:
Received 20 February 2016
Received in revised form 8 December 2016
Accepted 12 December 2016

Keywords:
Vertex-disjoint cycles
Partitions
Degree conditions
Specified vertex subsets

ABSTRACT

Let k be a positive integer, and let G be a graph of order $n \geq 3k$ and S be a set of k vertices of G. In this paper, we prove that if $\sigma_2(G) \geq n + k - 1 + \Delta(G[S])$, then G can be partitioned into k vertex-disjoint cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|V(C_i) \cap S| = 1$ for $1 \leq i \leq k$, and $|V(C_i)| = 3$ for $1 \leq i \leq k - 1 - \Delta(G[S])$ and $|V(C_i)| \leq 4$ for $k - \Delta(G[S]) \leq i \leq k - 1$, where $\sigma_2(G)$ denotes the minimum degree sum of two non-adjacent vertices in G and $\Delta(G[S])$ denotes the maximum degree of the subgraph of G induced by S. This is a common generalization of the results obtained by Dong (2010) and Chiba et al. (2010), respectively. In order to show the main theorem, we further give other related results concerning the degree conditions for the existence of k vertex-disjoint cycles in which each cycle contains a vertex in a specified vertex subset.

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [5]. Let G be a graph. We denote by $V(G)$, $E(G)$ and $\Delta(G)$ the vertex set, the edge set and the maximum degree of G, respectively. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S, and let $G - S = G[V(G) \setminus S]$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, we write $G - H$ instead of $G - V(H)$ for a subgraph H of G. We denote by $d_G(v)$ and $N_G(v)$ the degree and the neighborhood of a vertex v in G. The invariant $\sigma_2(G)$ is defined to be the minimum degree sum of two non-adjacent vertices of G, i.e., $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G), u \neq v, uv \notin E(G)\}$ if G is not complete; otherwise, let $\sigma_2(G) = +\infty$. In this paper, "partition" of a graph always means a partition of the vertex set.

In 1960, Ore [10] proved that every graph G of order $n \geq 3$ with $\sigma_2(G) \geq n$ is hamiltonian, which is well known in graph theory. As a generalization of this, we can consider σ_2 conditions for graphs to be partitioned into k vertex-disjoint cycles. In fact, Brandt, Chen, Faudree, Gould and Lesniak [1] proved that Ore's condition also implies that a graph G can be partitioned into k vertex-disjoint cycles, and such problems have been extensively studied in graph theory (for example, see [2,4,8,9,11]). In this paper, we consider σ_2 conditions for graphs to be partitioned into k vertex-disjoint cycles in which each cycle contains a vertex in a specified vertex subset.

In 2000, Egawa, Faudree, Győri, Ishigami, Schelp and Wang proved the following result, which is also a generalization of Ore's classical theorem. (They actually gave a stronger result by considering the existence of k (≥ 2) vertex-disjoint cycles passing through "specified edges", see [7] for more details.) Here, for a set S of k vertices of a graph G, we say that m ($\leq k$)

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http://dx.doi.org/10.1016/j.disc.2016.12.010
0012-365X/

cycles $C_1, \ldots, C_m$ in G are *admissible with respect to S* if they are vertex-disjoint and $|V(C_i) \cap S| = 1$ for each i with $1 \le i \le m$. We simply say that $C_1, \ldots, C_m$ are admissible if there is no fear of confusion.

Theorem A (*Egawa et al.* [7]). *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 4k - 1$ and S be a set of k vertices of G. If $\sigma_2(G) \ge n + 2k - 2$, then G can be partitioned into k admissible cycles, i.e., G contains k admissible cycles $C_1, \ldots, C_k$ satisfying $V(G) = \bigcup_{1 \le i \le k} V(C_i)$.*

Later, Dong (2010) showed that the σ_2 condition in Theorem A also guarantees the existence of admissible cycles with "short" lengths except at most one cycle.

Theorem B (*Dong* [6]). *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k vertices of G. If $\sigma_2(G) \ge n + 2k - 2$, then G can be partitioned into k admissible cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|C_i| \le 4$ for $1 \le i \le k - 1$.*

On the other hand, Chiba, Egawa and Yoshimoto (2010) considered the case where the specified set S is independent (i.e., $G[S]$ is edgeless), and they showed that "$\sigma_2(G) \ge n + 2k - 2$" and "$|C_i| \le 4$" in Theorem B can be replaced by "$\sigma_2(G) \ge n + k - 1$" and "$|C_i| = 3$", respectively.

Theorem C (*Chiba, Egawa and Yoshimoto* [3]). *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k independent vertices of G. If $\sigma_2(G) \ge n + k - 1$, then G can be partitioned into k admissible cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|C_i| = 3$ for $1 \le i \le k - 1$.*

The purpose of this paper is to prove a result which is a common generalization of Theorems B and C. Specifically, we prove the following theorem.

Theorem 1. *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k vertices of G. If $\sigma_2(G) \ge n + k - 1 + \Delta(G[S])$, then G can be partitioned into k admissible cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|C_i| = 3$ for $1 \le i \le k - 1 - \Delta(G[S])$, and $|C_i| \le 4$ for $k - \Delta(G[S]) \le i \le k - 1$.*

Since $\Delta(G[S]) \le k - 1$, Theorem 1 immediately implies Theorem B. Since $\Delta(G[S]) = 0$ if and only if S is independent, Theorem 1 also implies Theorem C. Therefore, Theorem 1 is stronger than Theorems B and C.

In order to show Theorem 1, we actually consider appropriate degree conditions guaranteeing the existence of k admissible cycles that partition a graph (see Corollary 4). For a graph G of order $n \ge k \ge 1$ and a vertex subset S of G with $|S| = k$, we now consider the following degree sum conditions (D1), (D2) and (D3), and give the following two results (Theorems 2 and 3).

(D1) $d_{G-(S \setminus \{s\})}(s) + d_G(v) \ge n + k - 1$ for $s \in S$ and $v \in V(G - S)$ with $sv \notin E(G)$.
(D2) $d_G(u) + d_G(v) \ge n + k - 1$ for $u, v \in V(G - S)$ with $u \ne v$ and $uv \notin E(G)$.
(D3) $d_{G-(S \setminus \{s\})}(s) + d_{G-(S \setminus \{s'\})}(s') \ge n - (k - 1) + 2t - 1$ for $s, s' \in S$ with $s \ne s'$ and $ss' \notin E(G)$, where t is an integer with $0 \le t \le k - 1$.

The first one is a result concerning the existence of k admissible cycles with short lengths.

Theorem 2. *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k vertices of G. Furthermore, let t be an integer with $0 \le t \le k - 1 - \Delta(G[S])$. If G satisfies* (D1), (D2) *and* (D3), *then G contains k admissible cycles $C_1, \ldots, C_k$ such that $|C_i| = 3$ for $1 \le i \le t$, and $|C_i| \le 4$ for $t + 1 \le i \le k$.*

The next result shows that the collection of the cycles in Theorem 2 can be transformed into a partition.

Theorem 3. *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k vertices of G. Furthermore, let t be an integer with $0 \le t \le k - 1$. Suppose that there exist k admissible cycles $C'_1, \ldots, C'_{k-1}, C'_k$ such that $|C'_i| = 3$ for $1 \le i \le t$, and $|C'_i| \le 4$ for $t + 1 \le i \le k$. If G satisfies* (D1) *and* (D2), *then G can be partitioned into k admissible cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|C_i| = 3$ for $1 \le i \le t$, and $|C_i| \le 4$ for $t + 1 \le i \le k - 1$.*

As an immediate corollary of Theorems 2 and 3, we can get the following.

Corollary 4. *Let k be an integer with $k \ge 1$, and let G be a graph of order $n \ge 3k$ and S be a set of k vertices of G. Furthermore, let t be an integer with $0 \le t \le k - 1 - \Delta(G[S])$. If G satisfies* (D1), (D2) *and* (D3), *then G can be partitioned into k admissible cycles $C_1, \ldots, C_{k-1}, C_k$ such that $|C_i| = 3$ for $1 \le i \le t$, and $|C_i| \le 4$ for $t + 1 \le i \le k - 1$.*

The degree conditions in Corollary 4 are best possible in some sense (see Remarks 1 and 2 in Section 2), and hence Corollary 4 shows that the appropriate degree conditions imply the existence of k admissible cycles with short lengths. In addition, the condition "$t \le k - 1 - \Delta(G[S])$" cannot be replaced by "$t \le k - 1$" (see Remark 3 in Section 2).

Note that if $\sigma_2(G) \ge n + k - 1 + \Delta(G[S])$, then it is easy to check that G satisfies (D1), (D2) and (D3) for $t \le k - 1 - \Delta(G[S])$, and hence we can obtain Theorem 1 by applying Corollary 4 as $t = k - 1 - \Delta(G[S])$. Therefore, it suffices to show that Corollary 4 holds, i.e., we have only to prove Theorems 2 and 3. To show the two theorems, in Section 2, we give two technical results (Theorems 5 and 6) and give other corollaries concerning the existence of k admissible cycles by using the two results. We will summarize our overall results in Section 2 (see Table 1).