Discrete Mathematics 340 (2017) 695–703

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

A recursive algorithm for trees and forests

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ARTICLE INFO

Article history:
Received 8 September 2016
Received in revised form 4 December 2016
Accepted 13 December 2016

Keywords:
Forests
Rooted trees
Bipartite trees
Tripartite trees
Plane trees
k-edge colored trees

ABSTRACT

Trees or rooted trees have been generously studied in the literature. A forest is a set of trees or rooted trees. Here we give recurrence relations between the number of some kind of rooted forest with k roots and that with $k+1$ roots on $\{1, 2, \ldots, n\}$. Classical formulas for counting various trees such as rooted trees, bipartite trees, tripartite trees, plane trees, k-ary plane trees, k-edge colored trees follow immediately from our recursive relations.

1. Introduction and notations

The famous Cayley's formula for counting trees states that the number of labeled trees on $[n]$ is n^{n-2}. Clarke [7] first gives a refined version for Cayley's formula by setting up a recurrence relation. Erdélyi and Etherington [11] gives a bijection between semilabeled trees and partitions. This bijection was also discovered by Haiman and Schmitt [14]. A general bijective algorithm was given by Chen [4]. Aigner and Ziegler's book [1] collected four different proofs of Cayley's formula. We refer the reader to [5,6,8,13,15,20] for several recent results on the enumeration of trees. The goal of this paper is to establish simple linear recurrences between certain forests with roots $1, \ldots, k$ and forests with roots $1, \ldots, k+1$, from which one can deduce several classical results on counting trees.

The set of forests of k rooted trees on $[n]$ with roots $1, \ldots, k$ is denoted by $\mathcal{F}_n^k$. Suppose $F \in \mathcal{F}_n^k$ and x is a vertex of F, the subtree rooted at x is denoted by F_x. We say that a vertex y of F is a *descendant* of x, if y is a vertex of F_x, i.e., x is on the path from the root of T to the vertex y, and is denoted by $y \prec x$.

For any edge $e = (x, y)$ of a tree T in a forest F, if y is a vertex of T_x, we call x the *father vertex* of e, y the *child vertex* of e, x the father of y, and y a child of x, sometimes we also say e is *out of* x. The degree of a vertex x in a rooted tree T is the number of children of x, and is denoted by $\deg_T(x)$, or $\deg_F(x)$. As usual, a vertex with degree zero is called a leaf.

An unrooted labeled tree will be treated as a rooted tree in which the smallest vertex is chosen as the root. Moreover, if $\mathcal{A}$ is a set of trees, then we will use $\mathcal{A}[P]$ to denote the subset of all elements of $\mathcal{A}$ satisfying the condition P.

2. The fundamental recursion

One of our main results is as follows.

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http://dx.doi.org/10.1016/j.disc.2016.12.012
0012-365X/

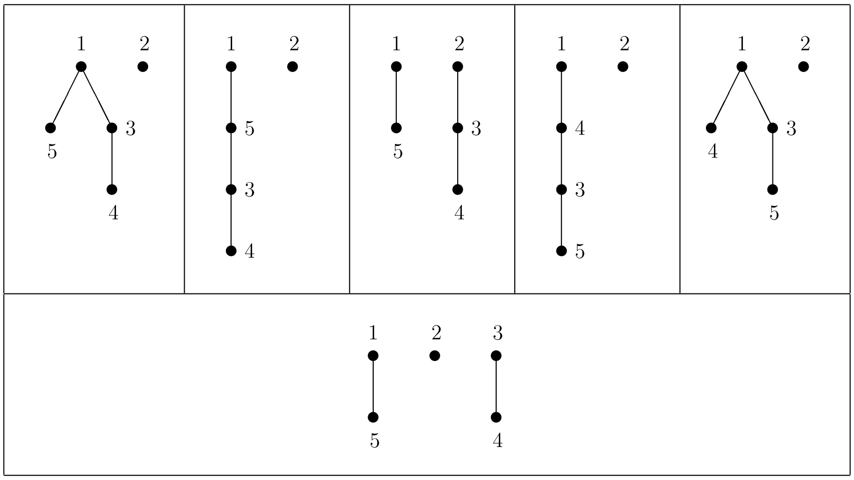

Fig. 1. Example of Theorem 2.1 for $n = 5$ and $k = 3$.

Theorem 2.1. *For* $2 \leqslant k \leqslant n-1$*, we have the following recurrence relation:*

$$|\mathcal{F}_n^{k-1}[n \prec 1]| = n|\mathcal{F}_n^k[n \prec 1]|. \tag{2.1}$$

Proof. Suppose $F \in \mathcal{F}_n^{k-1}[n \prec 1]$. First, remove the subtree F_k from F and add it to be a new tree in the forest. Second if n is not a descendant of 1 in the new forest, then n must be a descendant of k, and exchange labels of the vertices 1 and k. Thus, we obtain a forest $F' \in \mathcal{F}_n^k[n \prec 1]$.

Conversely, for a forest $F' \in \mathcal{F}_n^k[n \prec 1]$, we can attach F'_k to any vertex of the other trees in F as a subtree, or attach F'_1 to any vertex of F'_k as a subtree and exchange labels of the vertices 1 and k. The proof then follows from the fact that F' has n vertices altogether (see Fig. 1). $\square$

It is clear that

$$|\mathcal{F}_n^{n-1}[n \prec 1]| = 1. \tag{2.2}$$

We have the following corollaries.

Corollary 2.2 (*Cayley [3]*)**.** *The number of labeled trees on n vertices is* n^{n-2}.

Corollary 2.3 (*Cayley [3], Clarke [7]*)**.** *The number of rooted trees on* $n+1$ *vertices with a specific root and root degree k is* $\binom{n-1}{k-1} n^{n-k}$.

Proof. It follows from (2.1) and (2.2) that

$$|\mathcal{F}_n^k[n \prec 1]| = n^{n-k-1}.$$

Exchanging the labels of the vertices j and 1 for $1 < j \leqslant k < n$, we establish a bijection between $\mathcal{F}_n^k[n \prec 1]$ and $\mathcal{F}_n^k[n \prec j]$. Therefore,

$$|\mathcal{F}_n^k| = kn^{n-k-1},$$

from which one can see that the number of forests with n vertices and k trees is

$$\binom{n}{k} |\mathcal{F}_n^k| = \binom{n}{k} kn^{n-k-1} = \binom{n-1}{k-1} n^{n-k}. \quad \square$$

Remark. It is worth mentioning the third and fourth proofs of Cayley's formula in Aigner and Ziegler's book [1, Chapter 30]. The third proof in [1, Chapter 30], essentially due to Riordan [19] and Rényi [18], is as follows: Let $T_{n,k}$ denote the number of forests on $[n]$ consisting k trees where the vertices of $[k]$ appear in different trees. Consider such a forest F and suppose that 1 is adjacent to i vertices. Removing the vertex 1, we obtain a forest of $k-1+i$ trees. As we can reconstruct F by first fixing i, then selecting the i neighbors of 1, and then the forest $F \setminus 1$, this gives

$$T_{n,k} = \sum_{i=0}^{n-k} \binom{n-k}{i} T_{n-1,k-1+i},$$

from which we can prove Cayley's formula by induction on n.