Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Perspective

An introduction to the discharging method via graph coloring

Daniel W. Cranston [a], Douglas B. West [b,c,*]

[a] Virginia Commonwealth University, United States
[b] Zhejiang Normal University, China
[c] University of Illinois, United States

ARTICLE INFO

Article history:
Received 23 July 2013
Received in revised form 15 November 2016
Accepted 15 November 2016
Available online xxxx

Keywords:
Discharging method
Graph coloring
Planar graph
Edge-coloring
Maximum average degree
Light subgraph

ABSTRACT

We provide a "how-to" guide to the use and application of the Discharging Method. Our aim is not to exhaustively survey results proved by this technique, but rather to demystify the technique and facilitate its wider use, using applications in graph coloring as examples. Along the way, we present some new proofs and new problems.

1. Introduction

Arguments that can be phrased in the language of the Discharging Method have been used in graph theory for more than 100 years, though that name is much more recent. The most famous application of the method is the proof of the Four Color Theorem, stating that graphs embeddable in the plane have chromatic number at most 4. However, the method remains mysterious to many. Our aim is to explain its use and make the method more widely accessible. Although we mention many applications, including stronger versions of results proved here, cataloguing applications is not our goal. Borodin [22] presents a survey of applications of discharging to coloring of plane graphs.

Discharging is most commonly used as a tool in a two-pronged approach to inductive proofs, typically for sparse graphs. In this context, it is used to prove that a global sparseness hypothesis guarantees the existence of some desired local structure. The method has been applied to many types of problems (including graph embeddings and decompositions, spread of infections in networks, geometric problems, etc.). Nevertheless, we present only applications in graph coloring (where it has been used most often), in order to emphasize the discharging techniques.

In the simplest version, discharging just involves reallocation of vertex degrees in the context of a global bound on the average degree. We view each vertex as having an initial "charge" equal to its degree. To show that average degree less than b forces the presence of a desired local structure, we show that the absence of such a structure allows charge to be moved (via "discharging rules") so that the final charge at each vertex is at least b. This violates the hypothesis, and hence the desired structure must occur.

* Corresponding author.
E-mail addresses: dcranston@vcu.edu (D.W. Cranston), dwest@math.uiuc.edu (D.B. West).

http://dx.doi.org/10.1016/j.disc.2016.11.022

Please cite this article in press as: D.W. Cranston, D.B. West, An introduction to the discharging method via graph coloring, Discrete Mathematics (2016), http://dx.doi.org/10.1016/j.disc.2016.11.022

In an application of the resulting structure theorem, one shows that each such local structure is "reducible", meaning that it cannot occur in a minimal counterexample to the desired conclusion. This motivates the phrase "an unavoidable set of reducible configurations" to describe the overall process.

Definition 1.1. A *configuration* in a graph G can be any structure in G (often a specified sort of subgraph). A configuration is *reducible* for a graph property Q if it cannot occur in a minimal graph not having property Q. Let $d_G(v)$ or simply $d(v)$ denote the degree (number of neighbors) of vertex v in G, and let $\overline{d}(G)$ denote the average of the vertex degrees in G. *Degree charging* is the assignment to each vertex v of an "initial charge" equal to $d(v)$.

The notion of configuration is vague to permit use in various contexts. "Minimal" refers to some partial order on the graphs being considered; usually it is just minimality with respect to taking subgraphs, and the property Q is *monotone* (preserved by taking subgraphs).

Sparse local configurations aid in inductive proofs about coloring. For example, when $\overline{d}(G) < k$ with $k \in \mathbb{N}$, the pigeonhole principle guarantees a vertex with degree less than k in G. Also, when $d(v) < k$, a proper k-coloring of $G - v$ extends to a proper k-coloring of G. (A *k-coloring* is a function that assigns labels to vertices from a set of size k, a coloring of a graph G is *proper* if adjacent vertices have distinct colors, G is *k-colorable* if it admits a proper k-coloring, and the *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.)

In other words, vertices of degree less than k are reducible for the property $\chi(G) \le k$. However, guaranteeing such a vertex from the global bound $\overline{d}(G) < k$ does not need discharging. To illustrate how discharging works and interacts with reducibility, we consider another elementary example after introducing notation convenient for discussing vertex degrees.

Definition 1.2. A *j-vertex*, *j^+-vertex*, or *j^--vertex* is a vertex with degree equal to j, at least j, or at most j, respectively. A *j-neighbor* of v is a j-vertex that is a neighbor of v. We write $\delta(G)$ for the minimum and $\Delta(G)$ for the maximum of the vertex degrees in G.

Lemma 1.3. *If $\overline{d}(G) < 3$, then G has a 1^--vertex or a 2-vertex with a 5^--neighbor.*

Proof. We use degree charging; each vertex v starts with charge $d(v)$. Suppose that G has no 1^--vertex and that no 2-vertex in G has a 5^--neighbor. We move charge so that each vertex ends with charge at least 3. The 2-vertices need charge; 4^+-vertices can give charge.

Let each 2-vertex take $\frac{1}{2}$ from each neighbor. Now each 2-vertex has charge 3, since no two 2-vertices are adjacent. Vertices of degrees 3, 4, 5 lose no charge, since we assumed that no 2-vertex has a 5^--neighbor. Every 6^+-vertex v loses charge at most $\frac{1}{2}$ to each neighbor, leaving it with charge at least $d(v)/2$, which is at least 3 when $d(v) \ge 6$. Thus $\overline{d}(G) \ge 3$ when no 2-vertex has a 5^--neighbor. □

A 2-vertex with a 5^--neighbor is a local sparseness condition, somehow more sparse than a 2-vertex with high-degree neighbors. We first consider its use for edge-coloring. (A *k-edge-coloring* of a graph G assigns labels to edges from a set of size k; it is *proper* if incident edges have distinct colors, G is *k-edge-colorable* if it has a proper k-edge-coloring, and the *edge-chromatic number* $\chi'(G)$ is the least k such that G is k-edge-colorable.)

Here we phrase the reducibility statement in more generality. The *weight* of a subgraph H of a graph G is $\sum_{v \in V(H)} d_G(v)$; we sum the degrees in the full graph G.

Lemma 1.4. *An edge with weight at most $k + 1$ is a reducible configuration for the property of being k-edge-colorable.*

Proof. Let G be a graph having an edge e of weight at most $k + 1$. If the graph $G - e$ is k-edge-colorable, then a color is available to extend the coloring to e, because e is incident to a total of at most $k - 1$ other edges at its two endpoints. Thus a minimal graph G with $\chi'(G) > k$ cannot contain such a configuration. □

To complete an inductive proof of $\chi'(G) \le 6$ from Lemmas 1.3 and 1.4, we also need average degree less than 3 in subgraphs of G.

Definition 1.5. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum of the average degree over all subgraphs of G.

The application is now easy. Note that always $\chi'(G) \ge \Delta(G)$. In fact, Vizing's Theorem [73,117] states that always $\chi'(G) \le \Delta(G) + 1$, and distinguishing between $\chi'(G) = \Delta(G)$ and $\chi'(G) = \Delta(G) + 1$ is an important and difficult problem.

Theorem 1.6. *If $\mathrm{mad}(G) < 3$ and $\Delta(G) \ge 6$, then $\chi'(G) = \Delta(G)$.*

Proof. Fix an integer k at least 6. We prove more generally that if $\mathrm{mad}(G) < 3$ and $\Delta(G) \le k$, then $\chi'(G) \le k$. That is, among graphs with $\mathrm{mad}(G) < 3$ and $\Delta(G) \le k$ there is no minimal graph satisfying $\chi' > k$. Note that the hypotheses also hold in subgraphs.

We may discard isolated vertices. By Lemma 1.3, G then has a 1-vertex or has a 2-vertex with a 5^--neighbor. The edge incident to a 1-vertex has weight at most $\Delta(G) + 1$; an edge joining a 2-vertex to a 5^--neighbor has weight at most 7. In either case, the weight of this edge e is at most $k + 1$, and Lemma 1.4 implies that G is not a minimal graph satisfying $\chi'(G) > k$. Hence there is no minimal counterexample. □