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Odd properly colored cycles in edge-colored graphs

Gregory Gutin, Bin Sheng, Magnus Wahlström

Royal Holloway, University of London, TW20 0EX, Egham, Surrey, UK

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ABSTRACT

It is well-known that an undirected graph has no odd cycle if and only if it is bipartite. A less obvious, but similar result holds for directed graphs: a strongly connected digraph has no odd cycle if and only if it is bipartite. Can this result be further generalized to more general graphs such as edge-colored graphs? In this paper, we study this problem and show how to decide if there exists an odd properly colored cycle in a given edge-colored graph. As a by-product, we show how to detect if there is a perfect matching in a graph with even (or odd) number of edges in a given edge set.

1. Introduction

A graph G is *edge-colored* if each edge of G is assigned a color. (Edge-colorings can be arbitrary, not necessarily proper.) A cycle C in an edge-colored graph is *properly colored (PC)* if no pair of adjacent edges of C have the same color. It is not hard to see that PC cycles in edge-colored graphs generalize directed cycles (dicycles) in digraphs: in a digraph D replace every arc uv by an undirected path $ux_{uv}v$, where x_{uv} is a new vertex, and edges ux_{uv} and $x_{uv}v$ are of colors 1 and 2, respectively. Then PC cycles in the resulting graph correspond exactly to dicycles in D. Similarly, PC cycles generalize cycles in undirected graphs, e.g., by assigning every edge a distinct color.

One of the central topics in graph theory is the existence of certain kinds of cycles in graphs. In digraphs, it is not hard to decide the existence of any dicycle by simply checking whether a given digraph is acyclic [3]. The problem of existence of PC cycles in edge-colored graphs is less trivial. To solve the problem, we may use Yeo's theorem [20]: if an edge-colored graph G has no PC cycle then G contains a vertex z such that no connected component of $G - z$ is joined to z with edges of more than one color. Thus, we can recursively find such vertices z and delete them from G; if we end up with a trivial graph (containing just one vertex) then G has no PC cycles; otherwise G has a PC cycle. Clearly, the recursive algorithm runs in polynomial time.

One of the next natural questions is to decide whether a digraph has an odd (even, respectively) dicycle, i.e. a dicycle of odd (even) length, respectively. For odd dicycles we can employ the following well-known result (see, e.g., [3,8]): A strongly connected digraph is bipartite if and only if it has no odd dicycle. (Note that this result does not hold for non-strongly connected digraphs.) Thus, to decide whether a digraph D has an odd cycle, we can find strongly connected components of D and check whether all components are bipartite. This leads to a simple polynomial-time algorithm. The question of whether we can decide in polynomial time whether a digraph has an even dicycle, is much harder and was an open problem for quite some time till it was solved, in affirmative, independently by McCuaig, and Robertson, Seymour and Thomas (see [14]) who found highly non-trivial proofs.

In this paper we consider the problem of deciding the existence of an odd PC cycle in an edge-colored graph in polynomial time. We show that while a natural extension of the odd dicycle solution does not work, an algebraic approach using Tutte matrices and the Schwartz–Zippel lemma allows us to prove that there is a randomized polynomial-time algorithm for solving the problem. The existence of a deterministic polynomial-time algorithm for the odd PC cycle problem remains an open question, as does the existence of a polynomial-time algorithm for the even PC cycle problem.

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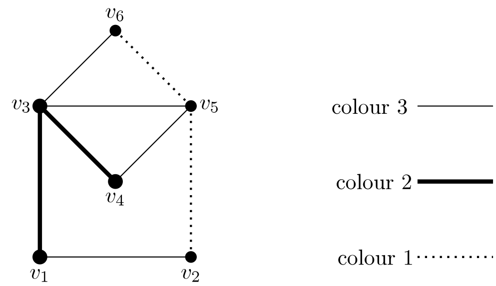

Fig. 1. Non-bipartite cyclic connected graph with no odd PC cycle.

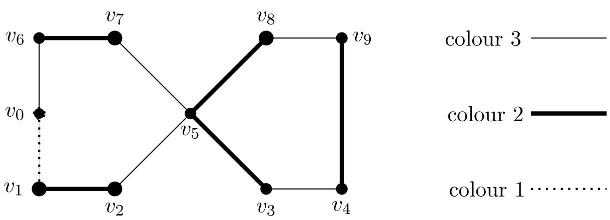

Fig. 2. An odd PC closed walk.

In this paper, we allow our graphs to have multiple edges (but no loops) and call them, for clarity, *multigraphs*. In edge-colored multigraphs, we allow parallel edges of different colors (there is no need to consider parallel edges of the same color). For an edge xy and a vertex v, we use $\chi(xy)$ and $\chi(v)$ to denote the color of xy and the set of colors of edges incident to v, respectively. All our graphs are undirected; as discussed above, edge-colored multigraphs properly generalize digraphs, even though they are undirected. For any other terminology and notation not provided here, we refer the readers to [3]. There is an extensive literature on PC paths and cycles: for a detailed survey of pre-2009 publications, see Chapter 16 of [3]; more recent papers include [1,5,9–11].

2. Graph-theoretical approaches

In this section, we consider some graph-theoretical approaches that have been successfully used for detecting odd cycles in (non-colored) directed and undirected graphs, and find that they are unlikely to work for detecting odd PC cycles in edge-colored graphs.

Recall that to solve the odd dicycle problem, in the previous section, we used the following result: A strongly connected digraph is bipartite if and only if it has no odd dicycle. It is not straightforward to generalize strong connectivity to edge-colored multigraphs. Indeed, color-connectivity,[1] introduced by Saad [15] under another name, does not appear to be useful to us as, in general, it does not partition vertices into components. Thus, we will use cyclic connectivity introduced by Bang-Jensen and Gutin [2] as follows. Let $P = \{H_1, \dots, H_p\}$ be a set of subgraphs of an edge colored multigraph G. The *intersection graph* $\Omega(P)$ of P has the vertex set P and the edge set $\{H_iH_j : V(H_i) \cap V(H_j) \neq \emptyset, 1 \leq i < j \leq p\}$. A pair x, y of vertices in an edge-colored multigraph H is *cyclic connected* if H has a collection of PC cycles $P = \{C_1, \dots, C_p\}$ such that x and y each belong to some cycles in P and $\Omega(P)$ is a connected graph. A maximal cyclic connected induced subgraph of G is called a *cyclic connected component* of G. Note that cyclic connected components partition the vertices of G. Also note that cyclic connectivity for digraphs, where dicycles are considered instead of PC cycles, coincides with strong connectivity. One could wonder whether every non-bipartite cyclic connected edge-colored graph has an odd PC cycle. Unfortunately, it is not true, see a graph H in Fig. 1. It is not hard to check that H is not bipartite and cyclic connected. It has even PC cycles, such as $v_1v_2v_5v_3v_1$, but no odd PC cycles.

Another natural idea is to find some odd PC closed walk first, and hope to find an odd PC cycle in it. Unfortunately, we cannot generate all possible PC closed walks in polynomial time, and moreover a PC closed walk does not necessarily contain an odd PC cycle, see the graph in Fig. 2. It contains an odd PC closed walk, but not an odd PC cycle.

3. Algebraic approach

In an edge-colored multigraph G, a vertex v is *monochromatic* if $|\chi(v)| = 1$. Let G' be the multigraph obtained from G by recursively deleting monochromatic vertices such that G' has no monochromatic vertex (for example, see Fig. 3). Following

[1] We will not define color-connectivity; an interested reader can find its definition in Sec. 16.6 of [3].

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