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Degree choosable signed graphs

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ABSTRACT

A signed graph is a graph in which each edge is labeled with $+1$ or -1. A (proper) vertex coloring of a signed graph is a mapping ϕ that assigns to each vertex $v \in V(G)$ a color $\phi(v) \in \mathbb{Z}$ such that every edge vw of G satisfies $\phi(v) \neq \sigma(vw)\phi(w)$, where $\sigma(vw)$ is the sign of the edge vw. For an integer $h \geq 0$, let $Z_{2h} = \{\pm 1, \pm 2, \ldots, \pm h\}$ and $Z_{2h+1} = Z_{2h} \cup \{0\}$. Following Máčajová et al. (2016), the chromatic number $\chi(G)$ of the signed graph G is the least integer k such that G admits a vertex coloring ϕ with $\mathrm{im}(\phi) \subseteq Z_k$. As proved in Máčajová et al. (2016), every signed graph G satisfies $\chi(G) \leq \Delta(G) + 1$ and there are three types of signed connected simple graphs for which equality holds. We will extend this Brooks' type result by considering graphs having multiple edges. We will also prove a list version of this result by characterizing degree choosable signed graphs. Furthermore, we will establish some basic facts about color critical signed graphs.

1. Introduction

This paper deals with the vertex coloring problem for signed graphs introduced by Zaslavsky [14–16] in the 1980s. Recently Máčajová, Raspaud and Škoviera [10] proved an extension of Brooks' theorem to signed simple graphs. Our aim is to characterize signed graphs that are degree choosable and to establish some basic properties of critical signed graphs.

Signed graphs

Signed graphs were first defined and investigated by Harary [5]. Throughout this paper, the term graph refers to a finite graph which may have multiple edges but no loops. A *signed graph* is a graph in which the edges are labeled by $+1$ or -1. So a signed graph is a triple $G = (V, E, \sigma)$, where $V = V(G)$ is the vertex set of G, $E = E(G)$ is the edge set of G, and $\sigma = \sigma_G$ is the sign mapping of G, i.e. $\sigma : E(G) \to \{-1, 1\}$. In order to make a clear distinction between a signed graph and its underlying graph, we shall use G° to denote the underlying graph of a signed graph G.

For a signed graph G we adopt the standard notations for graphs. For $X, Y \subseteq V(G)$, let $E_G(X, Y)$ be the set of all edges joining a vertex of X with a vertex of Y. Let $E_G[X] = E_G(X, X)$ be the set of all edges with both ends in X, and let $\partial_G X$ the set of all edges with exactly one end in X. If the meaning is clear we will frequently omit subscripts and brackets for the sake of readability. Thus, the degree of a vertex v in G is $d_G(v) = |\partial_G v|$, and the multiplicity of two distinct vertices v, w in G is $\mu_G(v, w) = |E_G(v, w)|$. As usual, we denote by $\Delta(G)$, $\delta(G)$ and $\mu(G)$ the *maximum degree*, the *minimum degree* and the *maximum multiplicity* of G, respectively. Thus, a simple signed graph is a signed graph G with $\mu(G) \leq 1$.

Switching and balance

An edge of a signed graph G is said to be *positive* or *negative* depending on whether its sign is +1 or -1. An *all positive signed graph* is a signed graph consisting only of positive edges; and an *all negative signed graph* is a signed graph consisting only of negative edges. A signed graph is all positive if and only if $\sigma = 1$, i.e., $\sigma(e) = 1$ for all $e \in E(G)$.

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If X is a vertex set of a signed graph G, then a new signed graph G' can be obtained by reversing the sign of each edge belonging to the coboundary $\partial_G X$. Then $G'^\circ = G^\circ$, $\sigma'(e) = -\sigma(e)$ for $e \in \partial_G X$ and $\sigma(e) = \sigma'(e)$ otherwise. We then say that G' is obtained from G by *switching* at X and write $G' = G^X$. Let X and Y be subsets of $V(G)$. Furthermore, let $\overline{X} = V(G) \setminus X$ and $X + Y = (X \cup Y) \setminus (X \cap Y)$. Then $G^X = G^{\overline{X}}$, $G^\varnothing = G$ and $(G^X)^Y = G^{(X+Y)}$. Two signed graphs G and G' are *switching equivalent*, written $G \equiv G'$, if there is a vertex set $X \subseteq V(G)$ such that $G' = G^X$. This obviously defines an equivalence relation for the class of signed graphs.

If G is a signed graph and H is a signed subgraph of G, that is, H° is a subgraph of G° and $\sigma_H = \sigma_G|_{E(H)}$, then

$$\sigma_G(H) = \prod_{e \in E(H)} \sigma_G(e)$$

is called the *sign product* of H. A signed graph G is called *balanced* if the sign product of each cycle of G is positive, otherwise it is called *unbalanced*. Clearly, if a signed graph G is balanced, then any signed graph switching equivalent to G is balanced, too. The following characterization of balanced graphs was obtained by Harary [5], proposition (c) was stated and proved by Zaslavsky [13], as well as, independently, by Sozański [11].

Theorem 1.1. *For a signed graph the following statements are equivalent:*

(a) *G is balanced.*
(b) *The vertex set of G is the disjoint union of two sets X and Y such that an edge of G is negative if and only if this edge belongs to $E_G(X, Y)$.*
(c) *G is switching equivalent to an all positive signed graph.*

If a signed graph G satisfies statement (b) of the above theorem, we say that G is a *balanced graph with parts* X and Y.

A signed graph G is *antibalanced* if the sign product of every even cycle of G is positive and the sign product of every odd cycle of G is negative. The *negation* of a signed graph G is the signed graph obtained from G by reversing the sign of all edges of G. Obviously, a signed graph is antibalanced if and only if its negation is balanced. So Harary's characterization of balanced graphs implies the following result.

Theorem 1.2. *For a signed graph the following statements are equivalent:*

(a) *G is antibalanced.*
(b) *The vertex set of G is the disjoint union of two sets X and Y such that an edge of G is positive if and only if this edge belongs to $E_G(X, Y)$.*
(c) *G is switching equivalent to an all negative signed graph.*

Chromatic number of signed graphs

Let G be a signed graph. A *coloring* of G is a mapping $\phi : V(G) \to \mathbb{Z}$ such that every edge $e \in E_G(v, w)$ satisfies $\phi(v) \neq \sigma(e)\phi(w)$. It is notable that if two vertices v and w of G are joined by a pair of differently signed parallel edges, then $|\phi(v)| \neq |\phi(w)|$. The above definition is due to Zaslavsky [14] and was mainly motivated by the following two simple observations. A coloring of an all positive signed graph is an ordinary vertex coloring of its underlying graph. Furthermore, if ϕ is a coloring of a signed graph G and $G' = G^X$ for a vertex set X of G, then the mapping ϕ', satisfying $\phi'(v) = -\phi(v)$ if $v \in X$ and $\phi'(v) = \phi(v)$ otherwise, is a coloring of G'. We denote the coloring ϕ' by ϕ^X.

A subset C of $\mathbb{Z}$ is called a *color set*. Given a color set C, let $-C = \{-c \mid c \in C\}$; in particular, C is called *symmetric* if $C = -C$. By $\sigma(e)C$ we mean C or $-C$ depending on whether $\sigma(e)$ is positive or negative. For an integer $h \geq 0$, let $Z_{2h} = \{\pm 1, \pm 2, \dots, \pm h\}$ and $Z_{2h+1} = Z_{2h} \cup \{0\}$. The *chromatic number* $\chi(G)$ of the signed graph G is the least integer k such that G admits a coloring ϕ with $\mathrm{im}(\phi) \subseteq Z_k$.

The above definition of the chromatic number of signed graphs is due to Máčajová, Raspaud and Škoviera [10]. They also established some basic facts about it. By the two observations about colorings of signed graphs it follows that switching equivalent signed graphs have the same chromatic number and the chromatic number of a balanced signed graph coincides with the chromatic number of its underlying graph. As proved in [10], if G is a signed graph, then

$$\chi(G) \leq 2\chi(G^\circ) - 1 \tag{1.1}$$

and there are signed simple graphs for which equality holds. We want to introduce a class of signed graphs for which equality holds in (1.1). If H is a simple graph, then we denote by $G = 2H$ the signed graph obtained from H by replacing every edge of H by a pair of differently signed parallel edges. Then we have the following result.

Theorem 1.3. *If $G = 2H$ for a simple graph H, then $\chi(G) = 2\chi(H) - 1$.*

Proof. Let $k = \chi(H)$ and $h = \chi(G)$. Then there is an ordinary vertex coloring ϕ of H using colors $0, 1, \dots, k-1$. Obviously, ϕ is a coloring of the signed graph G with $\mathrm{im}(\phi) \subseteq Z_{2k-1}$. Hence $\chi(G) \leq 2\chi(H) - 1$. Since $h = \chi(G)$, there exists a coloring ϕ of G with $\mathrm{im}(\phi) \subseteq Z_h$. Let X be the set $X = \{v \in V(G) \mid \phi(v) < 0\}$, then $G^X = G$ and so $\phi' = \phi^X$ is a coloring of G with $\mathrm{im}(\phi') \subseteq Z_h$ and $\phi'(v) \geq 0$ for all $v \in V(G)$. Then ϕ' is an ordinary vertex coloring of G and we obtain that $k \leq \frac{h+1}{2}$, i.e., $\chi(H) \geq 2\chi(G) - 1$. $\square$