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Combinatorial extensions of Terwilliger algebras and wreath products of association schemes

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ABSTRACT

We introduce the notion of the combinatorial extension of a Terwilliger algebra by a coherent algebra. By using this notion, we find a simple way to describe the Terwilliger algebras of certain coherent configurations as combinatorial extensions of simpler Terwilliger algebras. In particular, given an association scheme $\mathcal{S}$ and another association scheme $\mathcal{R}$ such that the Terwilliger algebra of $\mathcal{R}$ is isomorphic to a coherent algebra, we prove that the Terwilliger algebra of the wreath product $\mathcal{S} \wr \mathcal{R}$ is isomorphic to the combinatorial extension of the Terwilliger algebra of $\mathcal{S}$ by a coherent algebra. We also show that the Terwilliger algebra of the wreath product $\mathcal{W}$ of rank 2 association schemes can be expressed as the combinatorial extension of adjacency algebras of association schemes induced by the closed subsets of $\mathcal{W}$. As a direct consequence, we obtain simple conceptual explanations and alternative proofs of many known results on the structures of Terwilliger algebras of wreath products of association schemes.

1. Introduction

The Terwilliger algebra of a commutative association scheme was introduced by Terwilliger [17] in 1992. It is a finite-dimensional noncommutative semisimple $\mathbb{C}$-algebra generated by the Bose–Mesner algebra and its dual of the association scheme. It has had a great impact on the development of the theory of association schemes over the last two decades, and also has interesting applications in other areas (for example, see [14]). The algebraic structures and representations of Terwilliger algebras of various association schemes have been studied by many researchers (for example, see [1,3,7,9,11,13,19] and the references therein). This algebra is more often called the T-algebra nowadays. The T-algebra can be also defined for an arbitrary coherent configuration, and is a useful tool for the study of the coherent configuration as is for the association schemes. Usually the T-algebra of a coherent configuration (or an association scheme) is not isomorphic to the adjacency algebra of any coherent configuration (or association scheme). But it is important to note that there are many coherent configurations whose T-algebras are isomorphic to the adjacency algebras of certain coherent configurations (see Example 1.3 for more details). As a result, some algebraic tools can be developed and employed to understand the structures of certain coherent configurations better as we will see in what follows.

Our work has been inspired by the work of Muzychuk and Ponomarenko [12] who expressed a quasi-thin scheme as a direct sum of others, and described the structure of a quasi-thin scheme from those of factors. In this paper, we introduce the notion of the *combinatorial extension of a T-algebra by a coherent algebra*, and study the structure of the T-algebras arisen

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as combinatorial extensions of simpler T-algebras. We shall see that the structure of such T-algebra can be described from the structural information of the simpler T-algebra and the coherent algebra used as the factors of the extension.

We will first prove that the combinatorial extension of a T-algebra by a coherent algebra is an algebra (see Theorem 1.2), and investigate its central primitive idempotents (see (3.2)). The central primitive idempotents of the adjacency algebra of the wreath product of coherent configurations are also obtained (see Theorem 4.1). Given association schemes $\mathcal{S} = (\Omega, S)$ and $\mathcal{R} = (\Delta, R)$, the wreath product $\mathcal{S} \wr \mathcal{R} := (\Omega \times \Delta, S \wr R)$ is an association scheme. The T-algebra $\mathcal{T}_{(x,y)}(\mathcal{S} \wr \mathcal{R})$ of $\mathcal{S} \wr \mathcal{R}$ with respect to some $(x, y) \in \Omega \times \Delta$ was studied in [7,9,13]. If the T-algebra $\mathcal{T}_y(\mathcal{R})$ is coherent, then we will prove that $\mathcal{T}_{(x,y)}(\mathcal{S} \wr \mathcal{R})$ is isomorphic to the combinatorial extension of $\mathcal{T}_x(\mathcal{S})$ by a coherent algebra (see Theorem 1.4). When $\mathcal{R}$ is a thin scheme, or a quasi-thin scheme, or a rank 2 scheme, the central primitive idempotents of $\mathcal{T}_{(x,y)}(\mathcal{S} \wr \mathcal{R})$ are found in [7,9]. The computations employed in these works are rather intricated. Our Theorem 1.4 (and Theorem 4.1) provides an easy conceptual explanation and proof for the results in [7,9]. When both T-algebras $\mathcal{T}_x(\mathcal{S})$ and $\mathcal{T}_y(\mathcal{R})$ are coherent, it is proved in [13] that $\mathcal{T}_{(x,y)}(\mathcal{S} \wr \mathcal{R})$ is also coherent. In this case the standard basis of $\mathcal{T}_{(x,y)}(\mathcal{S} \wr \mathcal{R})$ is given in [13], too. These results in [13] are also direct consequences of Theorem 1.4.

Usually it is difficult to determine the structure of the T-algebra of a coherent configuration (or an association scheme). So far only for a few classes of association schemes, the non-principal primitive ideals of their T-algebras can be described in some ways. Among these classes are association schemes isomorphic to the wreath product of rank 2 association schemes. Let $\mathcal{S}$ be the wreath product of rank 2 association schemes. The T-algebra of $\mathcal{S}$ is studied in [3]. In particular, it is proved in [3] that $\mathcal{S}$ is triply regular (see [3, Theorem 3.4]), and the non-principal primitive ideals of the T-algebra of $\mathcal{S}$ are one-dimensional (see [3, Theorem 4.2]). By the use of the notion of the combinatorial extension, we are able to obtain further properties of the structure of the T-algebra of $\mathcal{S}$. We will show that the T-algebra of $\mathcal{S}$ is the combinatorial extension of the adjacency algebras of association schemes induced by the closed subsets of $\mathcal{S}$ (see Theorem 1.5). Using Theorem 1.5, we can easily obtain the explicit formulas for the central primitive idempotents of the T-algebra of $\mathcal{S}$ (see [15, Theorem 2.6] and Theorem 5.5 in Section 5). The results in [3] mentioned above are direct consequences of Theorems 1.5 and 5.5.

For the rest of this introductory section, we state some main results of the paper explicitly. We begin by recalling some necessary definitions and notation.

Let Ω be a finite set, and C a partition of $\Omega \times \Omega$. For any nonempty subset X of Ω, let $1_X := \{(x, x) : x \in X\}$, and for any $s \in C$, let $s^\top := \{(x, y) : (y, x) \in s\}$. Also for any $x \in \Omega$ and $s \in C$, let $xs := \{y \in \Omega : (x, y) \in s\}$. The pair $\mathfrak{C} := (\Omega, C)$ is called a *coherent configuration* over Ω if the following conditions (i)–(iii) are satisfied (see [8]).

(i) There is a subset C_0 of C which is a partition of 1_Ω.
(ii) For any $s \in C$, $s^\top$ is also in C.
(iii) Given $r, s, t \in C$, for any $(x, y) \in s$, the number $p^s_{tr} = |xt \cap yr^\top|$ does not depend on the choice of $(x, y) \in s$.

Let $\mathfrak{C} = (\Omega, C)$ be a coherent configuration. The elements of Ω and C are called the *points* and *basic relations* of $\mathfrak{C}$, respectively, and the numbers $|\Omega|$ and $|C|$ are called the *order* and *rank* of $\mathfrak{C}$, respectively. Also the numbers p^s_{tr} in (iii) are called the *intersection numbers*. The unique basic relation containing $(x, y) \in \Omega \times \Omega$ is sometimes denoted by $r(x, y)$. A nonempty subset $X \subseteq \Omega$ is called a *fiber* of $\mathfrak{C}$ if $1_X \in C$. Let $\mathrm{Fib}(\mathfrak{C})$ denote the set of all fibers of $\mathfrak{C}$. The set Ω is the disjoint union of the fibers of $\mathfrak{C}$. For any basic relation $s \in C$, there are unique fibers X and Y such that $s \subseteq X \times Y$. Moreover, the number $n_s := |xs|$, which does not depend on the choice of $x \in X$, is called the *valency* of s, and $n_s = p^{1_X}_{ss^\top}$. If $\mathfrak{C}$ has only one fiber, then it is called a *homogeneous coherent configuration* or an *association scheme* (or simply a *scheme*).

Let $M_\Omega(\mathbb{C})$ be the algebra of all $|\Omega| \times |\Omega|$ matrices over the complex numbers $\mathbb{C}$. The rows and columns of matrices in $M_\Omega(\mathbb{C})$ are indexed by the elements of Ω. For any subsets X and Y of Ω, let $I_X \in M_\Omega(\mathbb{C})$ be the diagonal matrix whose (x, x)-entry is 1 (if $x \in X$) or 0 (if $x \notin X$), and let $J_{X,Y} \in M_\Omega(\mathbb{C})$ be the matrix whose (x, y)-entry is 1 (if $(x, y) \in X \times Y$) or 0 (if $(x, y) \notin X \times Y$). In particular, if $X = \emptyset$, then $I_X = 0$ (the zero matrix), and if either $X = \emptyset$ or $Y = \emptyset$, then $J_{X,Y} = 0$. We will also simply write $J_{X,X}$ as J_X. A subalgebra of $M_\Omega(\mathbb{C})$ is called a *coherent algebra* if it is closed with respect to the matrix transpose and Hadamard multiplication $\circ$, and contains I_Ω and J_Ω (see [8]). Let $\mathfrak{C} = (\Omega, C)$ be a coherent configuration. For any $s \in C$, its adjacency matrix $A_s \in M_\Omega(\mathbb{C})$ is a (0, 1)-matrix whose (x, y)-entry is 1 (if $(x, y) \in s$) or 0 (if $(x, y) \notin s$). The $\mathbb{C}$-subspace of $M_\Omega(\mathbb{C})$ spanned by $\{A_s : s \in C\}$ is a coherent algebra, called the adjacency algebra of $\mathfrak{C}$, and denoted by $A(\mathfrak{C})$. Furthermore, the fibers of $\mathfrak{C}$ are also called the *fibers* of $A(\mathfrak{C})$.

Let Ω be a finite set, $\mathcal{A}$ a subalgebra of $M_\Omega(\mathbb{C})$, and $x \in \Omega$. For any $A \in \mathcal{A}$, let $D(A)_x \in M_\Omega(\mathbb{C})$ be the diagonal matrix whose (y, y)-entry is the (x, y)-entry of A. Let $D(\mathcal{A})_x := \{D(A)_x : A \in \mathcal{A}\}$. Then the subalgebra generated by $\mathcal{A}$ and $D(\mathcal{A})_x$ is called the *Terwilliger algebra* (*T-algebra*) of $\mathcal{A}$ with respect to x, and denoted by $\mathcal{T}_x(\mathcal{A})$. A nonempty subset $X \subseteq \Omega$ is called a *fiber* of $\mathcal{T}_x(\mathcal{A})$ if $1_X \in \mathcal{T}_x(\mathcal{A})$. In particular, if $\mathcal{A} = A(\mathfrak{C})$ is the adjacency algebra of a coherent configuration $\mathfrak{C} = (\Omega, C)$, then the T-algebra $\mathcal{T}_x(\mathcal{A})$ is also called the *T-algebra* of $\mathfrak{C}$ with respect to x, and also denoted by $\mathcal{T}_x(\mathfrak{C})$. Note that the fibers of $\mathcal{T}_x(\mathfrak{C})$ are xs, where $s \in S$ and $xs \neq \emptyset$.

Let Ω, Δ be two disjoint finite sets, and let $M_{\Omega,\Delta}(\mathbb{C})$ be the set of all $|\Omega| \times |\Delta|$ matrices over $\mathbb{C}$ whose rows and columns are indexed by the elements of Ω and Δ, respectively. Note that for any $A \in M_\Omega(\mathbb{C})$, $B \in M_\Delta(\mathbb{C})$, $U \in M_{\Omega,\Delta}(\mathbb{C})$, and $V \in M_{\Delta,\Omega}(\mathbb{C})$, the block matrix

$$\begin{pmatrix} A & U \\ V & B \end{pmatrix} \in M_{\Omega\cup\Delta}(\mathbb{C}).$$