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Maximum number of edges in claw-free graphs whose maximum degree and matching number are bounded

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ABSTRACT

We determine the maximum number of edges that a claw-free graph can have, when its maximum degree and matching number are bounded. This is a famous problem that has been studied on general graphs, and for which there is a tight bound. The graphs achieving this bound contain in most cases an induced copy of $K_{1,3}$, the claw, which motivates studying the question on claw-free graphs. Note that on general graphs, if one of the mentioned parameters is not bounded, then there is no upper bound on the number of edges. We show that on claw-free graphs, bounding the matching number is sufficient for obtaining an upper bound on the number of edges. The same is not true for the degree, as a long path is claw-free. We give exact tight formulas for both when only the matching number is bounded and when both parameters are bounded. We also construct claw-free graphs whose edge numbers match the given bounds.

1. Introduction

Extremal graph theory is an important field in discrete mathematics. It studies questions like how large or small some parameter of a graph can be under a given set of conditions [2,9]. An important problem in this respect is the following: Given a family $\mathcal{L}$ of forbidden subgraphs, find those graphs which contain no graph in $\mathcal{L}$ as a subgraph and have the maximum number of edges. In 1941, Turán [11] described the so-called Turán graphs on n vertices that do not contain the complete graph K_k, for a fixed $k < n$, and have the maximum number of edges. The Erdős–Stone theorem from 1946 [6] extends Turán's result by bounding the number of edges in a graph that does not have a fixed Turán graph as a subgraph.

A similar extremal question that dates back to 1960 is the following: *What is the maximum number of edges that a graph can have if its maximum degree is less than i and the size of a maximum matching of it is less than j, for two given integers i and j?* This question is a special case of a more general problem studied by Erdős and Rado [5], and it was first resolved by Chvátal and Hanson [4]. Later, Balachandran and Khare [1] gave a more structural proof and they also identified some graphs whose number of edges matches the given upper bound. In most cases, these graphs have connected components each of which is a star, i.e., $K_{1,k}$ for an appropriate integer k. This gives rise to the question what happens if the smallest star,[1] $K_{1,3}$, is forbidden as an induced subgraph. This restriction describes exactly the class of claw-free graphs.

In this paper we give an exact formula for the maximum number of edges that a claw-free graph can have, when its maximum degree and matching number are bounded. First we show that bounding only the matching number of a claw-free graph already results in bounded number of edges, and we give a formula for this number. For large enough given bound

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[1] Here we take as the smallest star, a star that is not also a path.

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on the maximum degree, this number is smaller than that for general graphs, and the gap increases with increasing allowed degree. If the degree bound is small enough, then we are able to give a formula that matches the formula for general graphs. In all cases, we also describe the claw-free graphs whose edge numbers match the given upper bounds.

2. Preliminaries

We work with undirected and simple graphs. Such a graph is denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges of G, with $|V| = n$. The *neighborhood* of a vertex is the set of all vertices adjacent to it. The maximum degree of a vertex in G is denoted by $\Delta(G)$. The size of a maximum matching in G is called its *matching number* and denoted by $\nu(G)$. G has a *perfect matching* if $\nu(G) = n/2$. A set X of vertices is an *independent set* if no pair of vertices in X are adjacent, whereas X is a *clique* if every pair of vertices in X are adjacent. A vertex in a connected graph is a *cut vertex* if removing it disconnects the graph.

The complete bipartite graph $K_{1,i-1}$ is called an *i-star*, and a 4-star ($K_{1,3}$) is called a *claw*. A vertex whose neighborhood contains an independent set of size 3 is called a *claw-center*. A graph is *claw-free* if it does not have a claw as an induced subgraph. Equivalently, a graph is claw-free if none of its vertices is a claw-center. The following observation is thus obvious; it will be used frequently in our proofs.

Observation 1. *If the neighborhood of a vertex v can be partitioned into two cliques, then v is not a claw-center.*

For a given graph class $\mathcal{C}$ and two given positive integers i and j, we define $\mathcal{M}_{\mathcal{C}}(i, j)$ to be the set of all graphs G in $\mathcal{C}$ satisfying $\Delta(G) < i$ and $\nu(G) < j$. A graph in $\mathcal{M}_{\mathcal{C}}(i, j)$ with the maximum number of edges is called *edge-extremal*. Thus the question that we are resolving in this paper is determining the number of edges in edge-extremal claw-free graphs. In the remainder, we assume that edge-extremal graphs have no isolated vertices since adding isolated vertices to a graph does not increase the number of edges. We let $\mathcal{GEN}$ denote the class of all graphs, and $\mathcal{CF}$ the class of claw-free graphs.

The following theorem summarizes the results of Balachandran and Khare [1]. For a positive integer i, they defined K'_i to be the graph obtained by removing a perfect matching from the complete graph K_i on i vertices, adding a new vertex v, and making v adjacent to $i - 1$ of the other vertices.

Theorem 2 (*[1]*). *The maximum number of edges in an edge-extremal graph in $\mathcal{M}_{\mathcal{GEN}}(i, j)$ is*

$$(i-1)(j-1) + \left\lfloor \frac{i-1}{2} \right\rfloor \left\lfloor \frac{j-1}{\lceil \frac{i-1}{2} \rceil} \right\rfloor.$$

A graph with this number of edges is obtained by taking the disjoint union of r copies of i-star and q copies of

$$\begin{cases} K_i & \text{if } i \text{ is odd} \\ K'_i & \text{if } i \text{ is even,} \end{cases}$$

where q is the largest integer such that $j - 1 = q \left\lceil \frac{i-1}{2} \right\rceil + r$ and $r \geq 0$.

In the next two sections we will give the corresponding numbers and extremal graphs for $\mathcal{M}_{\mathcal{CF}}(i, j)$. First, in the following section we show that bounding only the matching number is sufficient to be able to bound the number of edges of a claw-free graph.

3. Maximum number of edges in claw-free graphs whose matching number is bounded

We start with the following lemma, which does not hold for arbitrary graphs, in particular for stars.

Lemma 3. *For a connected claw-free graph G, $n \leq 2\nu(G) + 1$.*

Proof. Let G be a connected claw-free graph. Sumner [10] has shown that every connected claw-free graph with an even number of vertices has a perfect matching. Consequently, if n is even, then $\nu(G) = \frac{n}{2}$, and the result follows. If n is odd, then remove a vertex from G that is not a cut vertex. Such a vertex exists by the results of Chartrand and Zhang [3]. The remaining graph is connected, claw-free, and has an even number of vertices; thus it has a perfect matching whose size is $(n - 1)/2$. Consequently, $\nu(G) \geq (n - 1)/2$, and the proof is complete. □

We now describe the edge-extremal claw-free graphs when the given bound on the maximum degree is large enough.

Theorem 4. *If $i \geq 2j$, then K_{2j-1} is the unique edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$, resulting in $(2j - 1)(j - 1)$ edges.*

Proof. Since K_{2j-1} is claw-free, $\nu(K_{2j-1}) = j-1 < j$, and $\Delta(K_{2j-1}) = 2j-2 \leq i-2 < i$, we have that $K_{2j-1} \in \mathcal{M}_{\mathcal{CF}}(i, j)$. Let us show that K_{2j-1} is the edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$. When $i \geq 2j$, if an edge-extremal graph in $\mathcal{M}_{\mathcal{CF}}(i, j)$ is connected, then by Lemma 3, it can have at most $2j - 1$ vertices. The graph that obtains the maximum number of edges on a given set of vertices is unique; it is the complete graph K_{2j-1}, and it has $(2j - 1)(j - 1)$ edges. Now assume that, for $i \geq 2j$, there