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Ramsey numbers of trees and unicyclic graphs versus fans

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ABSTRACT

The generalized Ramsey number $R(H, K)$ is the smallest positive integer n such that for any graph G with n vertices either G contains H as a subgraph or its complement $\overline{G}$ contains K as a subgraph. Let T_n be a tree with n vertices and F_m be a fan with $2m+1$ vertices consisting of m triangles sharing a common vertex. We prove a conjecture of Zhang, Broersma and Chen for $m \geq 9$ that $R(T_n, F_m) = 2n - 1$ for all $n \geq m^2 - m + 1$. Zhang, Broersma and Chen showed that $R(S_n, F_m) \geq 2n$ for $n \leq m^2 - m$ where S_n is a star on n vertices, implying that the lower bound we show is in some sense tight. We also extend this result to unicyclic graphs UC_n, which are connected graphs with n vertices and a single cycle. We prove that $R(UC_n, F_m) = 2n - 1$ for all $n \geq m^2 - m + 1$ where $m \geq 18$. In proving this conjecture and extension, we present several methods for embedding trees in graphs, which may be of independent interest.

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1. Introduction

Given two graphs H and K, the generalized Ramsey number $R(H, K)$ is the smallest positive integer n such that for any graph G with n vertices, either G contains H as a subgraph or the complement $\overline{G}$ of G contains K as a subgraph. When both H and K are complete graphs, $R(H, K)$ is the classical Ramsey number. Because classical Ramsey numbers are difficult to determine, Chvátal and Harary proposed to study generalized Ramsey numbers of graphs other than complete graphs through a series of papers in 1972 and 1973 [5–7].

Generalized Ramsey numbers have since been well studied for a variety of graphs, including trees and fans. Chvátal determined the Ramsey number of trees versus complete graphs, showing that $R(T_n, K_m) = (n-1)(m-1)+1$ for positive integers m and n [4]. Burr, Erdős, Faudree, Rousseau and Schelp determined the Ramsey number of large trees versus odd cycles, showing that $R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 756m^{10}$ [3]. Recently, we showed this result is also true for smaller trees satisfying $n \geq 25m$ [1]. Salman and Broersma determined the Ramsey number of paths versus fans, finding $R(P_n, F_m)$ for various ranges of n and m [11]. Shi determined the Ramsey number of cycles versus fans, showing that $R(C_n, F_m) = 2n - 1$ for all $n > 3m$ [12]. In [9], Li and Rousseau proved an upper bound on the Ramsey number of fans versus complete graphs, showing

$$R(F_m, K_n) \leq (1 + o(1))\frac{n^2}{\log n}.$$

A survey of Ramsey numbers and related lower bounds can be found in [10].

There have also been general lower bounds shown to hold for Ramsey numbers. In 1981, Burr proved the following lower bound in terms of the chromatic number $\chi(G)$ of a graph G and its chromatic surplus $s(G)$ –the minimum number of vertices in a color class over all proper vertex colorings of G using $\chi(G)$ colors.

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Theorem 1 (*Burr [2]*). *If H is a connected graph with n vertices and $s(K)$ is the chromatic surplus of the graph K, then for $n \geq s(K)$ we have*

$$R(H,K) \geq (n-1)(\chi(K)-1)+s(K).$$

The Ramsey numbers of trees versus odd cycles, of cycles versus fans and of trees versus complete graphs determined by Burr et al., Shi and Chvátal, respectively, achieve Burr's lower bound. Note that for a fan or odd cycle K, $\chi(K)=3$ and $s(K)=1$ and thus Theorem 1 implies that $R(T_n, F_m) \geq 2n-1$ for all m and $n \geq s(K)=1$. This lower bound can be seen directly by considering the complete bipartite graph $K_{n-1,n-1}$. Since $K_{n-1,n-1}$ is triangle-free, it does not contain F_m as a subgraph. Furthermore, $\overline{K_{n-1,n-1}}$ consists of two connected components of size $n-1$ and thus does not contain T_n as a subgraph.

In 2015, Zhang, Broersma and Chen showed that the lower bound from Theorem 1 is tight for large trees and stars versus fans, proving the following two theorems. Here, S_n denotes a star on n vertices consisting of an independent set of $n-1$ vertices all adjacent to a single vertex.

Theorem 2 (*Zhang, Broersma, Chen [13]*). *$R(T_n, F_m)=2n-1$ for all integers m and $n \geq 3m^2-2m-1$.*

Theorem 3 (*Zhang, Broersma, Chen [13]*). *$R(S_n, F_m)=2n-1$ for all integers $n \geq m^2-m+1$ and $m \neq 3,4,5$, and this lower bound is the best possible. Moreover, $R(S_n, F_m)=2n-1$ for $n \geq 6m-6$ and $m=3,4,5$.*

Because it is generally believed that $R(T_n, G) \leq R(S_n, G)$ for any graph G, Zhang, Broersma and Chen made the following conjecture based on Theorem 3.

Conjecture 1 (*Zhang, Broersma, Chen [13]*). *$R(T_n, F_m)=2n-1$ for all integers $m \geq 6$ and $n \geq m^2-m+1$.*

Theorem 3 yields that if $n \leq m^2-m$ then $R(S_n, F_m) \geq 2n$, implying $n \geq m^2-m+1$ is the best achievable lower bound on n in terms of m over which $R(T_n, F_m)=2n-1$ is true [13]. In this paper, we prove Conjecture 1 for the case $m \geq 9$. Specifically, we prove the following theorem.

Theorem 4. *$R(T_n, F_m)=2n-1$ for all $n \geq m^2-m+1$ for $m \geq 9$.*

In [13], Zhang, Broersma and Chen also determined $R(T_n, K_{\ell-1}+mK_2)$ as a corollary of Theorem 2. Here, mG denotes the union of m vertex-disjoint copies of G and G_1+G_2 is the graph obtained by joining every vertex of G_1 to every vertex of G_2 in $G_1 \cup G_2$. Zhang, Broersma and Chen identify $R(T_n, K_{\ell-1}+mK_2)$ for $n \geq 3m^2-2m-1$ by induction on ℓ, using Theorem 2 as a base case. Their induction argument remains valid when Theorem 4 is used as the base case, yielding the following updated version of their corollary.

Corollary 1 (*Zhang, Broersma, Chen [13]*). *$R(T_n, K_{\ell-1}+mK_2)=\ell(n-1)+1$ for $\ell \geq 2$ and $n \geq m^2-m+1$ where $m \geq 9$.*

We also extend Theorem 4 from trees to unicyclic graphs. Let UC_n denote a particular connected graph with n vertices and a single cycle—or equivalently a connected graph with n vertices and n edges. We prove the following result.

Theorem 5. *$R(UC_n, F_m)=2n-1$ for all $n \geq m^2-m+1$ for $m \geq 18$.*

Note that Theorem 5 implies Theorem 4 as a corollary in the case $m \geq 18$. Despite this, we present our proofs of these two theorems separately because our approach to Theorem 4 motivates our proof of Theorem 5 and because we require a sufficiently different approach and more careful analysis to prove Theorem 4 for $9 \leq m < 18$. The next section provides the notation and key lemmas that will be used in the proofs of Theorems 4 and 5. In the two subsequent sections, we prove Theorems 4 and 5.

2. Preliminaries and lemmas

We first provide the notation we will adopt on proving Theorems 4 and 5. Let G be any simple graph. Here, $d_X(v)$ denotes the degree of a vertex v in the set $X \subseteq V(G)$ in G and $\overline{d_X}(v)$ denotes the degree of v in X in the complement graph $\overline{G}$. Similarly, $N_X(v)$ and $\overline{N_X}(v)$ denote the sets of neighbors of v in the set X in G and $\overline{G}$, respectively. It is clear that $d_X(v)+\overline{d_X}(v)=|X|$ for any $X \subseteq V(G)$ not containing v and that $d_X(v)=|N_X(v)|$ and $\overline{d_X}(v)=|\overline{N_X}(v)|$. We also extend this notation to $N_X(Y)$ and $\overline{N_X}(Y)$ for sets $Y \subseteq V(G)$ disjoint from X. When the set X is omitted, it is implicitly $V(G)$ where the graph G is either clear from context or explicitly stated. We denote the maximum and minimum degrees of a graph G as $\Delta(G)$ and $\delta(G)$, respectively. When G is bipartite and connected, we let the sets $A(G)$ and $B(G)$ denote the partite sets of G with $V(G)=A(G) \cup B(G)$ and $|A(G)| \geq |B(G)|$. In particular, this implies that $|A(G)| \geq |V(G)|/2$. For a tree T, we let $L(T)$ denote the set of leaves of T. Also note that if T is a tree then since T is bipartite, $A(T)$ and $B(T)$ are well-defined.

We now prove two lemmas that will be used throughout the proofs of Theorems 4 and 5. The first is a structural lemma concerning the vertices of degree two in trees and will be crucial to our methods for embedding trees.

Lemma 1. *Given a tree T and a subset $F \subseteq V(T)$, there is a set D satisfying:*

(1) *$D \subseteq A(T)$ and $F \cap D = \emptyset$;*
(2) *each $v \in D$ satisfies $d_T(v)=2$;*