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Colorful paths for 3-chromatic graphs

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ABSTRACT

In this paper, we prove that every 3-chromatic connected graph, except C_7, admits a 3-vertex coloring in which every vertex is the beginning of a 3-chromatic path with 3 vertices. It is a special case of a conjecture due to S. Akbari, F. Khaghanpoor, and S. Moazzeni stating that every connected graph G other than C_7 admits a $\chi(G)$-coloring such that every vertex of G is the beginning of a colorful path (i.e. a path on $\chi(G)$ vertices containing a vertex of each color). We also provide some support for the conjecture in the case of 4-chromatic graphs.

1. Introduction

In this paper, we deal with oriented and non-oriented graphs. When it is not specified, graphs are supposed to be non-oriented. Notation not given here are consistent with [6]. The vertex set of a graph or an oriented graph G is denoted by $V(G)$ and its edge set (or arc set) by $E(G)$. [1] Classically, for a vertex x of a graph G, a vertex y with $\{x, y\} \in E(G)$ is called a *neighbor* of x. The set of all the neighbors of x, denoted by $N_G(x)$, is the *neighborhood* of x in G. In the oriented case, an *out-neighbor* (resp. *in-neighbor*) of a vertex x of an oriented graph G is a vertex y with $xy \in E(G)$ (resp. $yx \in E(G)$). Similarly, the set of all the out-neighbors (resp. in-neighbors) of x in G, denoted by $N_G^+(x)$ (resp. $N_G^-(x)$) is the *out-neighborhood* (resp. *in-neighborhood*) of x in G.

In a graph G, we denote by $x_1 \dots x_{\ell+1}$ the *path* of length ℓ on the distinct vertices $\{x_1, \dots, x_{\ell+1}\}$ with edges $\{x_1, x_2\}$, $\{x_2, x_3\}, \dots, \{x_\ell, x_{\ell+1}\}$. We denote also by $x_1 \dots x_\ell x_1$ the *cycle* C_ℓ of length ℓ on the distinct vertices $\{x_1, \dots, x_\ell\}$ with edges $\{x_1, x_2\}, \dots, \{x_{\ell-1}, x_\ell\}, \{x_\ell, x_1\}$. Classically, these notions are extended to oriented graphs, where the arcs x_ix_{i+1} replace the edges $\{x_i, x_{i+1}\}$ (computed modulo ℓ for the oriented cycle C_ℓ). In the whole paper, the structures we consider are not induced, except if explicitly stated.

A *k-(proper) coloring* of a graph G is a mapping $c : V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k for which G admits a k-coloring and thus, we say that G is a $\chi(G)$-*chromatic graph*. For a k-coloring of a graph G, a *rainbow path* of G is a path whose vertices have all distinct colors. Given a $\chi(G)$-coloring of G, a rainbow path on $\chi(G)$ vertices is a *colorful path*. In particular a colorful path is transversal to the set of colors (*i.e.* it has a non-empty intersection with every color class). Finding structures transversal to a partition of the ground set is a general problem in combinatorics. Examples arise from Steiner Triple Systems (see [9]), systems of representatives (see [1]) or extremal graph theory (see [11]). Rainbow and colorful paths have been extensively studied in the last few years, see for instance [2,3,8,12] and [13]. In this paper, we concentrate on a conjecture of S. Akbari, F. Khaghanpoor and S. Moazzeni raised in [2] (also cited in [7]).

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[1] Throughout the paper, we use notation xy to indicate the (oriented) arc from x to y, while $\{x, y\}$ designates the (non-oriented) edge between x and y.

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Conjecture 1 (*S. Akbari, F. Khaghanpoor and S. Moazzeni [2]*). *Every connected graph G other than C_7 admits a $\chi(G)$-coloring such that every vertex of G is the beginning of a colorful path.*

Conjecture 1 holds for 1-chromatic graphs and 2-chromatic graphs. Indeed in connected bipartite graphs, every vertex is connected to a vertex of another color. The classical proof of Gallai–Roy Theorem also shows that in any $\chi(G)$-coloring of a graph G, there exists at least one colorful path (see [6], for instance). Furthermore, much more is known concerning this conjecture which, though recent, has already received attention. In [3], S. Akbari, V. Liaghat, and A. Nikzad proved that Conjecture 1 is true for the graphs G having a complete sub-graph of size $\chi(G)$. They also proved that every graph G admits a $\chi(G)$-coloring such that every vertex is the beginning of a rainbow path on $\lfloor \frac{\chi(G)}{2} \rfloor$ vertices. This result was improved by M. Alishahi, A. Taherkhani and C. Thomassen in [4], who showed that we can obtain rainbow paths on $\chi(G) - 1$ vertices.

In this paper, we give further evidence for Conjecture 1, and prove it for 3-chromatic graphs.

Theorem 2. *Every connected 3-chromatic graph G other than C_7 admits a 3-coloring such that every vertex of G is the beginning of a colorful path.*

The proof of Theorem 2 uses an auxiliary oriented graph built from a coloring of the instance graph. This oriented graph was already used in [3]. In Section 2, we recall its definition and strengthen the results known about it to obtain some useful lemmas. In Section 3, we use these tools to derive the proof of Theorem 2. Finally, in Section 4, we conclude the paper with some remarks and open questions. In particular, we prove that Conjecture 1 is true for 4-chromatic graphs containing a cycle of length four.

2. Preliminaries

In this section, $G = (V, E)$ is a connected graph and c is a proper coloring of G with $\chi(G)$ colors. Here, G is not necessarily 3-chromatic and, for short, we write χ instead of $\chi(G)$. In the following, we will consider modifications of colors and all these modifications have to be understood modulo χ. As defined in [3], the oriented graph D_c has vertex set V and ab is an arc of D_c if $\{a, b\}$ is an edge of G and the color of b equals the color of a plus one (this oriented graph was first introduced in [10,14]). A colorful path starting at the vertex x is called a *certifying path for x*. A colorful path $x_1 \dots x_\chi$ is *forward* (resp. *backward*) if for every $i \in \{1, \dots, \chi - 1\}$ we have $c(x_{i+1}) = c(x_i) + 1 \bmod \chi$ (resp. $c(x_{i+1}) = c(x_i) - 1 \bmod \chi$). Note that a forward (resp. backward) certifying path for a vertex x is an oriented path in D_c on χ vertices starting (resp. ending) at x.

An *initial section* of D_c is a subset X of V such that there is no arc of D_c entering into X (i.e. from $V(G) \setminus X$ to X). The *initial recoloring* of X consists of reducing the color used on each vertex in X by one. Using initial recolorings it is possible to prove some basic facts on the existence of colorful paths. The remaining of this section is devoted to these results (note that Lemmas 3 and 4 are mentioned in [3], but we recall here their short original proofs for the sake of completeness).

Lemma 3 (*S. Akbari et al. [3]*). *An initial recoloring of an initial section is still a proper coloring.*

Proof. Let c be a coloring of G and X an initial section of D_c. We denote by c' the coloring of G obtained after the initial recoloring of X. Let x and y be two adjacent vertices. If both x and y are not in X, we have $c'(x) = c(x) \neq c(y) = c'(y)$. If both x and y are in X, we have $c'(x) = c(x) - 1 \neq c(y) - 1 = c'(y)$. So, by symmetry we may assume that $x \notin X$ and $y \in X$. Since X is an initial section, there is no arc from x to y in D_c and then we have $c(x) \neq c(y) - 1$. Thus we have $c'(x) = c(x) \neq c(y) - 1 = c'(y)$. □

We will intensively use Lemma 3 to prove Theorem 2, and so, without referring it precisely. Notice that when performing an initial recoloring on an initial section X, we remove from D_c all the arcs leaving X and possibly add some arcs entering into X (the arcs xy with $\{x, y\} \in E(G)$, $x \notin X$, $y \in X$ and $c(x) = c(y) - 2$). Moreover, we do not create any arc leaving X. Indeed suppose by contradiction that an arc xy is created with $x \in X$ and $y \notin X$, then in the original coloring c, we must have $c(x) = c(y)$, contradicting c being proper. The other arcs, standing inside or outside X remain unchanged. Similarly, a subset X of vertices is a *terminal section* of D_c if there is no arc leaving X (i.e. from X to $V(G) \setminus X$). The *terminal recoloring* of X consists in adding one to the color of the vertices of X. As for the initial recoloring, this coloring is still proper. Note also that, when performing a terminal recoloring of X, we remove from D_c all the arcs entering into X and possibly add some arcs leaving X (the arcs xy with $\{x, y\} \in E(G)$, $x \in X$, $y \notin X$ and $c(x) = c(y) - 2$).

Two colorings c and c' are *identical* on X if $c(x) = c'(x)$ for all $x \in X$. Concerning identical coloring, we have the following.

Lemma 4 (*S. Akbari et al. [3]*). *Let c be a χ-coloring of G and X be a subset of vertices of G. There exists a χ-coloring c' of G identical to c on X such that every vertex is the beginning of an oriented path of $D_{c'}$ which ends in X.*

Proof. Let c'' be a χ-coloring of G identical with c on X. We define $Y_{c''}$ as the set of vertices of G which are the beginning of an oriented path in $D_{c''}$ ending in X. The path can have length 0, *i.e.* X is included in $Y_{c''}$. Now, we choose c' a χ-coloring of G identical with c on X with an associated set $Y_{c'}$ of maximal cardinality. Let us prove that $Y_{c'} = V$. Otherwise, by definition, $Y_{c'}$ is an initial section of $D_{c'}$, and so that $V \setminus Y_{c'}$ is a terminal section of $D_{c'}$. Denote by c_t the terminal recoloring of $V \setminus Y_{c'}$. As $X \subset Y_{c'}$, c_t is also identical to c on X. Moreover, the arcs from $Y_{c'}$ to $V(G) \setminus Y_{c'}$ of $D_{c'}$ are not anymore in D_{c_t} and the only