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# Forbidden pairs for spanning (closed) trails

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ABSTRACT

In Faudree and Gould (1997), the authors determined all pairs of connected graphs $\{H_1, H_2\}$ such that any connected $\{H_1, H_2\}$-free graph has a spanning path (cycle), *i.e.*, hamiltonian path (cycle). In this paper, we consider a similar problem and determine all pairs of forbidden subgraphs guaranteeing the existence of spanning (closed) trails of connected graphs. Our results show that although the forbidden pairs for the existence of spanning trails are the same as the existence of spanning paths, the forbidden pairs for the existence of spanning closed trails (supereulerian) are much different from those for the existence of spanning cycles (hamiltonian).

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## 1. Introduction

Throughout this paper, all graphs considered are finite, undirected and simple except the case when the cycle $C_2$. We follow [11] for the most common graph-theoretical terminology and notation not defined here. For a connected graph $H$, a graph $G$ is said to be $H$-free if $G$ does not contain an induced subgraph which is isomorphic to $H$. More generally, for a set of connected graphs $\mathcal{H}$, $G$ is said to be $\mathcal{H}$-free if $G$ is $H$-free for every $H \in \mathcal{H}$. The set $\mathcal{H}$ is often referred to as forbidden subgraphs. If $\mathcal{H} = \{H\}$, then we simply say that $G$ is $H$-free and $G$ is **claw-free** if $H = K_{1,3}$. We call $\mathcal{H}$ a **forbidden pair** if $|\mathcal{H}| = 2$.

A **trail** of a graph $G$, denoted by $T$, is a sequence $T := v_0e_1v_1 \cdots v_{l-1}e_lv_l$, whose terms are alternately vertices and edges of $G$ such that $v_{i-1}$ and $v_i$ are the ends of $e_i$ $(1 \le i \le l)$ and its edge terms are distinct. A **spanning trail** of a graph $G$ is a trail containing all vertices of $G$. A path or a cycle that contains all vertices of $G$ is called a **hamiltonian path** or a **hamiltonian cycle**, respectively. A graph is called **hamiltonian** or **traceable** if it contains a hamiltonian cycle or hamiltonian path, respectively.

Duffus et al. [6] presented the earliest forbidden subgraph result on hamiltonian and traceable properties, as shown in Theorem 1. For positive integers $i, j, k$ with $i \ge j \ge k \ge 0$, let $N_{i,j,k}$ be the graph (see Fig. 1) obtained by identifying end vertices of three disjoint paths of lengths $i, j, k$ to three vertices of a triangle.

**Theorem 1** (*Duffus, Gould and Jacobson [4]*). *Let G be a $\{K_{1,3}, N_{1,1,1}\}$-free graph. Then*

(i) *if G is connected, then G is traceable;*
(ii) *if G is 2-connected, then G is hamiltonian.*

After this, the pairs of forbidden subgraphs of a graph have attracted much more attentions by scholars. In 1982, Gould and Jacobson [7] investigated the relationship between a pair of forbidden subgraphs and hamiltonian properties of graphs.

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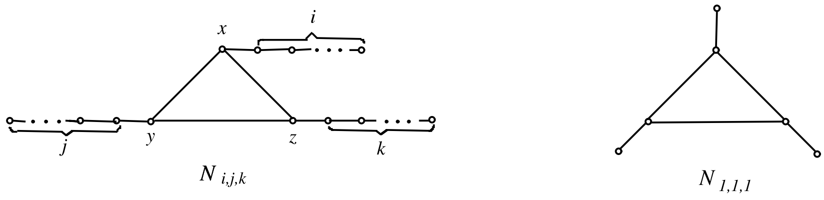


**Fig. 1.** The graphs $N_{i,j,k}$ and $N_{1,1,1}$.

In 1997, Faudree and Gould [5] extended Duffus et al.'s result, and characterized all pairs of forbidden subgraphs that force the existence of a hamiltonian path in connected graphs of sufficiently large order.

**Theorem 2** (*Faudree and Gould [5]*). *Let $R$ and $S$ be connected graphs ($R, S \neq P_3$) and let $G$ be a connected graph. Then $G$ is $\{R, S\}$-free implies $G$ is traceable if, and only if, $R = K_{1,3}$ and $S$ is one of the graphs $C_3, P_4, N_{1,0,0}, N_{1,1,0}$ or $N_{1,1,1}$.*

Recently, the research of forbidden subgraphs becomes more popular again. In this paper, motivated by Theorem 2, we determine all pairs of forbidden subgraphs for a spanning trail of a connected graph, and we obtain the following theorem.

**Theorem 3.** *Every connected $\{K_{1,3}, N_{1,1,1}\}$-free graph has a spanning trail.*

Note that although Theorem 3 can be deduced immediately by Theorem 1(i) or Theorem 2, we provide its complete proof in Section 2. Since $\{H_0, H_1\}$-free graph has some property $P$ and $H_2$ is an induced subgraph of $H_1$, $\{H_0, H_2\}$-free graph has also the same property $P$. It is not difficult to see that $P_4$ is an induced subgraph of $N_{1,1,1}$, so we have the following corollary.

**Corollary 4.** *If $G$ is a connected $\{K_{1,3}, P_4\}$-free graph, then $G$ has a spanning trail.*

On the basis of Theorem 3, we consider the reversed problem: Which pairs $\{H_1, H_2\}$ imply a connected graph $G$ having a spanning trail? We notice that Theorem 3 shows the pair $\{K_{1,3}, N_{1,1,1}\}$ is one such pair. It is easy to see that if $H$ is any induced subgraph of $N_{1,1,1}$, then the pair $\{K_{1,3}, H\}$ can also solve our problem. Since $C_3, P_3, P_4, N_{1,0,0}$ and $N_{1,1,0}$ are the induced subgraphs of $N_{1,1,1}$, each of them may play the role of $H$. However, $G$ is $P_3$-free implies that $G$ is a complete graph, and then $G$ has a spanning trail. In fact, $P_3$ is an induced subgraph of $K_{1,3}$, $P_3$-free implies $K_{1,3}$-free, so forbidding $P_3$ alone implies that $G$ has a spanning trail. Thus, we remove it from consideration in the pairs of forbidden subgraphs. Then we obtain the following two results:

**Theorem 5.** *Let $X, Y(\neq P_3)$ be connected graphs of order at least 3, and let $G$ be a connected graph. Then $G$ is $\{X, Y\}$-free implies $G$ has a spanning trail if, and only if, $X = K_{1,3}$ and $Y$ is one of the following: $C_3, P_4, N_{1,0,0}, N_{1,1,0}$ or $N_{1,1,1}$.*

**Theorem 6.** *Let $H$ and $G$ be connected graphs. Then $G$ is $H$-free implies $G$ has a spanning trail if, and only if, $H = P_3$.*

In the following, we use $K'_{2,3}$ denote the graph obtained from $K_{2,3}$ by subdividing exactly one edge of $K_{2,3}$. The following result characterizes the pairs of forbidden subgraphs for a 2-connected graph to have a spanning closed trail, *i.e.*, **supereulerian**. Because every supereulerian nontrivial graph is 2-edge-connected, its nontrivial block should be supereulerian and hence we may consider its nontrivial block. Since every nontrivial block is 2-connected, we may always assume those graphs in consideration are 2-connected.

**Theorem 7.** *Let $X$ and $Y$ be connected graphs ($X, Y \notin \{P_2, P_3\}$) and let $G$ be a 2-connected graph of order $n$ such that $G \notin \{K_{2,3}, K'_{2,3}\}$. Then $G$ is $\{X, Y\}$-free implies $G$ is supereulerian if, and only if, either $X = K_{1,4}$ and $Y$ is a subgraph of $P_5$, or $X = K_{1,3}$ and $Y$ is one of the graphs $P_4, P_5, P_6, P_7, C_3, N_{0,0,1}, N_{0,0,2}, N_{0,0,3}, N_{0,0,4}, N_{0,1,1}, N_{0,1,2}, N_{0,1,3}, N_{1,1,1}, N_{1,1,2}$ or $N_{1,1,3}$.*

Before closing this section, we need some additional notations, which will be used in our proofs later. For a graph $G$ and for $\emptyset \neq S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by the set $S$. We denote by $N_G(x)$ and $d_G(x)$ the neighborhood and degree of a vertex $x$ in a graph $G$, respectively. We use $P_n$, $C_n$ and $K_n$ to denote the path, cycle and complete graph with $n$ vertices, respectively. If $v_0$ and $v_l$ are end vertices of a trail $T$, then we call $v_0$ and $v_l$ the origin and terminus vertices of $T$, respectively, and we denote this trail $T$ by $v_0Tv_l$.

## 2. Proofs of Theorems 3, 5 and 6

In this section, we provide the proofs of Theorems 3, 5 and 6.

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