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Complete graph immersions in dense graphs

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ABSTRACT

In this article we consider the relationship between vertex coloring and the immersion order. Specifically, a conjecture proposed by Abu-Khzam and Langston in 2003, which says that the complete graph with t vertices can be immersed in any t-chromatic graph, is studied.

First, we present a general result about immersions and prove that the conjecture holds for graphs whose complement does not contain any induced cycle of length four and also for graphs having the property that every set of five vertices induces a subgraph with at least six edges.

Then, we study the class of all graphs with independence number less than three, which are graphs of interest for Hadwiger's Conjecture. We study such graphs for the immersion-analog. If Abu-Khzam and Langston's conjecture is true for this class of graphs, then an easy argument shows that every graph of independence number less than 3 contains $K_{\lceil \frac{n}{2} \rceil}$ as an immersion. We show that the converse is also true. That is, if every graph with independence number less than 3 contains an immersion of $K_{\lceil \frac{n}{2} \rceil}$, then Abu-Khzam and Langston's conjecture is true for this class of graphs. Furthermore, we show that every graph of independence number less than 3 has an immersion of $K_{\lceil \frac{n}{3} \rceil}$.

1. Introduction

Vertex coloring has been a very important topic in graph theory. The usual goal, and the one considered here, is to color every vertex of a graph such that adjacent vertices get different colors. The *chromatic number* of a graph G, denoted as $\chi(G)$, is the minimum number of colors required to color its vertices. If $\chi(G) = t$, then we say that G is *t-chromatic*.

It has been suspected for a long time that if a graph cannot be colored with $t - 1$ colors, then it has to somehow contain the complete graph K_t with t vertices. At some point in the 40s, Hajós [17] conjectured that the relation of containment was the topological order. This conjecture is true for $t \leq 4$ [10], but false for $t \geq 7$ [4]. It remains open for $t \in \{5, 6\}$. In 1943 Hadwiger [16] suggested that the containment had to be the minor order, i.e. he conjectured that every t-chromatic graph contains K_t as a minor. It was shown that Hadwiger's conjecture holds for $t = 5$ [27] and $t = 6$ [23], but it remains uncertain whether or not the conjecture is true for $t \geq 7$.

In this article we study a different order, the immersion order, which is defined by lifts of edges. A *lift* of two (adjacent) edges uv and vw, with $u \neq w$ and $uw \notin E(G)$, consists of deleting uv and vw, and adding the edge uw. And a graph H is *immersed* in a graph G if H can be obtained from G by performing lifts of edges and deleting vertices and/or edges. We denote this by $H \preccurlyeq_i G$. We also say that G contains an *immersion* of H. This definition is equivalent [1] to the existence of an injective function $\phi : V(H) \to V(G)$ such that:

1. For every $uv \in E(H)$, there is a path in G, denoted as P_{uv}, which connects $\phi(u)$ and $\phi(v)$.
2. The paths $\{P_{uv} : uv \in E(H)\}$ are pairwise edge-disjoint.

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If the paths P_{uv} are internally disjoint from $\phi(V(H))$, then we say that the immersion is *strong*. We call the vertices in $\phi(V(H))$ the *corner vertices* of the immersion.

Clearly topological containment implies immersion containment (strong immersion containment, actually). However, the minor order and the immersion order are not comparable. The immersion order, although initially much less studied than the minor and topological orders, has received a large amount of attention recently [3,12–15,18,28]. In fact, Robertson and Seymour extended their proof of Wagner's famous conjecture [21], to prove that the immersion order is a well-quasi-order [22].

In analogy to Hadwiger and Hajós' conjectures, Lescure and Meyniel [19] conjectured the following.

Lescure and Meyniel's Conjecture. *If $\chi(G) \geq t$, then G contains a strong immersion of K_t.*

Independently, Abu-Khzam and Langston [1] proposed a weaker statement.

Conjecture 1 (*Abu-Khzam and Langston*)**.** *If $\chi(G) \geq t$, then K_t is immersed in G.*

Since Hajós' conjecture holds for $t \leq 4$, Abu-Khzam and Langston's conjecture is true for $t \leq 4$, as topological order is just a particular case of immersion order.

Each graph G with $\chi(G) = t$ must contain a *t-critical* subgraph, i.e., a graph $\widetilde{G}$ such that $\chi(\widetilde{G}) = t$ and $\chi(H) < t$ for every proper subgraph H of $\widetilde{G}$. Furthermore, it is easy to see that every t-critical graph must have minimum degree at least $t - 1$. Using this fact, DeVos, Kawarabayashi, Mohar and Okamura [9] resolved Abu-Khzam and Langston's conjecture for small values of t.

Theorem 1.1 (*[9]*)**.** *Let $f(k)$ be the smallest integer such that every graph of minimum degree at least $f(k)$ contains an immersion of K_k. Then $f(k) = k - 1$ for $k \in \{5, 6, 7\}$.*

For $k \geq 8$, however, $f(k) \geq k$ [7,8], i.e. $\delta(G) \geq k - 1$ does not guarantee an immersion of K_k in G.

Theorem 1.1 solves Abu-Khzam and Langston's conjecture for very small values of t. We are interested here in the other end of the spectrum, where t is close to the number of vertices. So we restrict our attention to classes of graphs which are quite dense. We already know some properties about dense graphs, such that if a graph has $2cn^2$ edges, then it contains a strong immersion of the complete graph on at least c^2n vertices [8].

One very special case of dense graphs are the complete multipartite graphs. We prove the following result.

Theorem 1.2. *Let G be a complete multipartite graph of $k \geq 2$ classes with s vertices each. Then G has a strong immersion of H, where,*

$$H = \begin{cases} K_{(k-1)s+1} & \text{if } s \text{ is even} \\ K_{(k-1)s} & \text{if } s \neq 1 \text{ and } s \text{ is odd} \\ K_k & \text{if } s = 1. \end{cases}$$

We will call a graph (k, s)-*dense* if every set of k vertices induces a subgraph with at least s edges. We prove the following two results.

Theorem 1.3. *Every (5,6)-dense graph G contains a strong immersion of $K_{\chi(G)}$.*

Theorem 1.4. *Any graph G whose complement has no induced cycle of length four contains a strong immersion of $K_{\chi(G)}$.*

Finally, we focus on the study of a special class of graphs, the graphs G that have no independent set of size three, or equivalently, whose independence number $\alpha(G)$ is at most 2. This class of graphs has been extensively studied in an attempt to solve Hadwiger's conjecture (see [2,5,6,20]). It is for this reason that we are interested in Abu-Khzam and Langston's conjecture restricted to these graphs. Abu-Khzam and Langston's conjecture restricted to that class reads as follows.

Conjecture 2. *Any graph G with $\alpha(G) \leq 2$ contains an immersion of $K_{\chi(G)}$.*

If $\alpha(G) \leq 2$, then in any vertex coloring of G, every color class, being an independent set, has at most two vertices, which implies that $\chi(G) \geq \frac{n}{2}$. Abu-Khzam and Langston's conjecture would thus imply that G must contain an immersion of $K_{\lceil \frac{n}{2} \rceil}$. The latter gives rise to a new conjecture.

Conjecture 3. *Any graph G with $\alpha(G) \leq 2$ contains an immersion of $K_{\lceil \frac{n}{2} \rceil}$.*

We just saw that Conjecture 2 implies Conjecture 3. However, the two conjectures are actually equivalent. Following ideas from [20] we show the next result.

Theorem 1.5. *Conjectures 2* and *3 are equivalent.*

A weaker version of Conjecture 3 is shown, namely the following result.