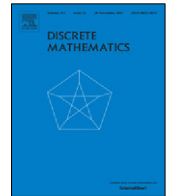




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Note

Stars on trees

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ABSTRACT

For a positive integer r and a vertex v of a graph G , let $\mathcal{I}_G^{(r)}(v)$ denote the set of independent sets of G that have exactly r elements and contain v . Motivated by a problem of Holroyd and Talbot, Hurlbert and Kamat conjectured that for any r and any tree T , there exists a leaf z of T such that $|\mathcal{I}_T^{(r)}(v)| \leq |\mathcal{I}_T^{(r)}(z)|$ for each vertex v of T . They proved the conjecture for $r \leq 4$. We show that for any integer $k \geq 3$, there exists a tree T_k that has a vertex x such that x is not a leaf of T_k , $|\mathcal{I}_{T_k}^{(r)}(z)| < |\mathcal{I}_{T_k}^{(r)}(x)|$ for any leaf z of T_k and any integer r with $5 \leq r \leq 2k + 1$, and $2k + 1$ is the largest integer s for which $\mathcal{I}_{T_k}^{(s)}(x)$ is non-empty.

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1. Introduction

We shall use small letters such as x to denote non-negative integers or elements of a set, capital letters such as X to denote sets or graphs, and calligraphic letters such as \mathcal{F} to denote *families* (that is, sets whose members are sets themselves). The set $\{1, 2, \dots\}$ of positive integers is denoted by \mathbb{N} . For any $m, n \in \mathbb{N}$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$, and we abbreviate $[1, n]$ to $[n]$. For a set X , the family $\{A \subseteq X : |A| = r\}$ of r -element subsets of X is denoted by $\binom{X}{r}$. If $x \in X$ and \mathcal{F} is a family of subsets of X , then the family $\{F \in \mathcal{F} : x \in F\}$ is denoted by $\mathcal{F}(x)$ and is called a *star* of \mathcal{F} . Arbitrary sets are assumed to be finite.

A graph G is a pair (X, \mathcal{Y}) , where X is a set, called the *vertex set* of G , and \mathcal{Y} is a subset of $\binom{X}{2}$ and is called the *edge set* of G . The vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. An element of $V(G)$ is called a *vertex* of G , and an element of $E(G)$ is called an *edge* of G . We may represent an edge $\{v, w\}$ by vw . If vw is an edge of G , then we say that v is *adjacent* to w (in G). A vertex v of G is a *leaf* of G if it is adjacent to only one vertex of G .

If H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that G *contains* H .

If v_1, v_2, \dots, v_n are the distinct vertices of a graph G with $E(G) = \{v_i v_{i+1} : i \in [n-1]\}$, then G is called a (v_1, v_n) -*path* or simply a *path*.

A graph G is a *tree* if $|V(G)| \geq 2$ and G contains exactly one (v, w) -path for every $v, w \in V(G)$ with $v \neq w$.

Let G be a graph. A subset I of $V(G)$ is an *independent set* of G if $vw \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G^{(r)}$ denote the family of all independent sets of G of size r . An independent set J of G is *maximal* if $J \not\subseteq I$ for each independent set I of G such that $I \neq J$. The size of a smallest maximal independent set of G is denoted by $\mu(G)$.

Hurlbert and Kamat [11] conjectured that for any $r \geq 1$ and any tree T , there exists a leaf z of T such that $\mathcal{I}_T^{(r)}(z)$ is a star of $\mathcal{I}_T^{(r)}$ of maximum size.

Conjecture 1.1 ([11, Conjecture 1.25]). *For any $r \geq 1$ and any tree T , there exists a leaf z of T such that $|\mathcal{I}_T^{(r)}(v)| \leq |\mathcal{I}_T^{(r)}(z)|$ for each $v \in V(T)$.*

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Hurlbert and Kamat [11] also showed that the conjecture is true for $r \leq 4$. In the next section, we show that for any $k \geq 3$, there exists a tree T_k that has a vertex x such that x is not a leaf of T_k , $|\mathcal{I}_{T_k}^{(r)}(z)| < |\mathcal{I}_{T_k}^{(r)}(x)|$ for any leaf z of T_k and any $r \in [5, 2k+1]$, and $2k+1$ is the largest integer s for which $\mathcal{I}_{T_k}^{(s)}(x)$ is non-empty. At the time of finalizing this paper, it came to the author's attention that this was proved for $k \geq r^2$ by Baber [1], remarkably using the same construction for T_k ; however, the proof presented here differs in that it provides a partitioning argument by which only the structural difference between two competing stars is quantified, and by which the full result is obtained.

Conjecture 1.1 was motivated by a problem of Holroyd and Talbot [8, 10]. A family \mathcal{A} is *intersecting* if every two sets in \mathcal{A} intersect. We say that $\mathcal{I}_G^{(r)}$ has the *star property* if at least one of the largest intersecting subfamilies of $\mathcal{I}_G^{(r)}$ is a star of $\mathcal{I}_G^{(r)}$. Holroyd and Talbot introduced the problem of determining whether $\mathcal{I}_G^{(r)}$ has the star property for a given graph G and an integer $r \geq 1$. The Holroyd–Talbot (HT) Conjecture [10, Conjecture 7] claims that $\mathcal{I}_G^{(r)}$ has the star property if $\mu(G) \geq 2r$. By the classical Erdős–Ko–Rado Theorem [5], the HT Conjecture is true if G has no edges. The HT Conjecture has been verified for certain graphs [3, 4, 6, 7, 9–13]. It is also verified in [2] for any graph G with $\mu(G)$ sufficiently large depending on r ; this is the only result known for the case where G is a tree that is not a path (the problem for paths is solved in [9]), apart from the above-mentioned result of Hurlbert and Kamat, and the fact that $\mathcal{I}_G^{(r)}$ may not have the star property for certain values of r (indeed, if G is the tree $(\{0\} \cup [n], \{\{0, i\} : i \in [n]\})$ and $2 \leq n/2 < r < n$, then $\mathcal{I}_G^{(r)} = \binom{[n]}{r}$ and $\binom{[n]}{r}$ is intersecting). One of the difficulties in trying to establish the star property lies in determining a largest star. The following counterexample to **Conjecture 1.1** indicates that the problem for trees is more difficult than is hoped.

2. The result

Let $x_0 = 0, x_1 = 1$, and $x_2 = 2$. For any positive integer k , let $y_i = 2 + i$ for each $i \in [2k]$, let $z_i = 2k + 2 + i$ for each $i \in [2k]$, and let T_k be the graph whose vertex set is

$$\{x_0, x_1, x_2\} \cup \{y_i : i \in [2k]\} \cup \{z_i : i \in [2k]\}$$

and whose edge set is

$$\{x_0x_1, x_0x_2\} \cup \{x_1y_i : i \in [k]\} \cup \{x_2y_i : i \in [k+1, 2k]\} \cup \{y_iz_i : i \in [2k]\}.$$

We remark that for the purpose of our result, the vertices $x_0, x_1, x_2, y_1, \dots, y_{2k}, z_1, \dots, z_{2k}$ of T_k could be any $4k+3$ distinct objects (that is, not necessarily the integers $0, 1, \dots, 4k+2$). What is important is that x_0 is adjacent to x_1 and x_2 , x_1 is adjacent to the k vertices y_1, \dots, y_k , x_2 is adjacent to the k vertices y_{k+1}, \dots, y_{2k} , y_i is adjacent to z_i for each $i \in [2k]$, and there are no other adjacencies.

Theorem 2.1. *Let k be a positive integer.*

- (a) *The graph T_k is a tree, and the leaves of T_k are z_1, \dots, z_{2k} .*
- (b) *The largest integer s such that $\mathcal{I}_{T_k}^{(s)}(x_0) \neq \emptyset$ is $2k+1$.*
- (c) *If $k \geq 3$, then $|\mathcal{I}_{T_k}^{(r)}(z)| < |\mathcal{I}_{T_k}^{(r)}(x_0)|$ for any leaf z of T_k and any $r \in [5, 2k+1]$.*

Proof. (a) is straightforward.

Let $G = T_k$. Let $Y = \{y_i : i \in [2k]\}$ and $Z = \{z_i : i \in [2k]\}$.

We have $\{x_0\} \cup Z \in \mathcal{I}_G^{(2k+1)}(x_0)$. Suppose that S is a set in $\mathcal{I}_G^{(s)}(x_0)$. Thus, $S \setminus \{x_0\} \in \binom{Y \cup Z}{s-1}$ and $|(S \setminus \{x_0\}) \cap \{y_i, z_i\}| \leq 1$ for each $i \in [2k]$. Therefore, $s-1 \leq 2k$, and hence $s \leq 2k+1$. Hence (b).

Suppose $k \geq 3$ and $r \in [5, 2k+1]$. Let $\mathcal{J} = \mathcal{I}_G^{(r)}$. Let $\mathcal{E} = \{I \in \mathcal{J} : x_0, z_1 \in I\}$. We will compare the number $|\mathcal{J}(x_0) \setminus \mathcal{E}|$ of sets in \mathcal{J} that contain x_0 but not z_1 , with the number $|\mathcal{J}(z_1) \setminus \mathcal{E}|$ of sets in \mathcal{J} that contain z_1 but not x_0 . Let

$$\begin{aligned} \mathcal{A}_1 &= \{I \in \mathcal{J}(x_0) : y_1 \in I\}, \\ \mathcal{A}_2 &= \{I \in \mathcal{J}(x_0) : y_1, z_1 \notin I\}, \\ \mathcal{B}_1 &= \{I \in \mathcal{J}(z_1) : x_0 \notin I, x_1 \in I, x_2 \notin I\}, \\ \mathcal{B}_2 &= \{I \in \mathcal{J}(z_1) : x_0 \notin I, x_1 \notin I, x_2 \in I\}, \\ \mathcal{B}_3 &= \{I \in \mathcal{J}(z_1) : x_0 \notin I, x_1, x_2 \in I\}, \\ \mathcal{B}_4 &= \{I \in \mathcal{J}(z_1) : x_0, x_1, x_2 \notin I\}. \end{aligned}$$

We have $\mathcal{J}(x_0) = \mathcal{E} \cup \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{J}(z_1) = \mathcal{E} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Since $y_1z_1 \in E(G)$, $\{y_1, z_1\} \not\subseteq I$ for each $I \in \mathcal{J}$. Thus, \mathcal{E} , \mathcal{A}_1 , and \mathcal{A}_2 are pairwise disjoint, and hence

$$|\mathcal{J}(x_0)| = |\mathcal{E}| + |\mathcal{A}_1| + |\mathcal{A}_2|. \quad (1)$$

Since \mathcal{E} , \mathcal{B}_1 , \mathcal{B}_2 , \mathcal{B}_3 , and \mathcal{B}_4 are pairwise disjoint,

$$|\mathcal{J}(z_1)| = |\mathcal{E}| + |\mathcal{B}_1| + |\mathcal{B}_2| + |\mathcal{B}_3| + |\mathcal{B}_4|. \quad (2)$$

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