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Note

Minimum supports of eigenfunctions of Hamming graphs☆

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ABSTRACT

We prove that the minimum Hamming weight (the number of nonzeros) of an eigenfunction of the Hamming graph $H(n, q)$ corresponding to the second largest eigenvalue is $2(q-1)q^{n-2}$.

1. Introduction

Many combinatorial configurations (for example, perfect codes, latin squares and hypercubes, combinatorial designs and their q-ary generalizations — subspace designs) can be defined as an eigenfunction on a graph with some discrete restrictions. The study of these configurations often leads to the question about the minimum possible difference between two configurations from the same class (it is often related with bounds of the number of different configurations; for example, see [1,2,4–6,8]). Since the symmetric difference of these two configurations is also an eigenfunction, this question is directly related to the minimum cardinality of the support (the set of nonzero) of an eigenfunction with given eigenvalue. In more details, these connections are described in [4], where the minimum cardinality of the support of an eigenfunction of the Grassmann graph with the smallest eigenvalue was found. This paper is devoted to the problem of finding the minimum cardinality of the support of eigenfunctions in the Hamming graphs $H(n, q)$. Currently, this problem is solved only for $q = 2$ (see [3,5]). In particular, the problem is related to the question of the minimum difference of two q-ary perfect codes corresponding to the eigenvalue -1. In this paper we find the minimum cardinality of the support of eigenfunctions in the Hamming graphs with the second largest eigenvalue $n(q-1)-q$ and describe the set of functions with the minimum cardinality of the support.

2. Basic definitions

Let $\Sigma_q = \{0, 1, \ldots, q-1\}$. The *Hamming distance* $d(x, y)$ between vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from Σ_q^n is the number of positions i such that $x_i \neq y_i$. The *Hamming graph* $H(n, q)$ is the graph whose vertex set is Σ_q^n and two vertices are adjacent if the Hamming distance between them equals 1. The set of neighbors of a vertex x is denoted by $N(x)$. It is well known that the set of eigenvalues of the adjacency matrix of $H(n, q)$ is $\{\lambda_m = n(q-1) - qm \mid m = 0, 1, \ldots, n\}$. A function $f : \Sigma_q^n \longrightarrow \mathbb{R}$ is called an *eigenfunction* of $H(n, q)$ corresponding to an eigenvalue λ if

$$\lambda f(x) = \sum_{y \in N(x)} f(y).$$

In what follows we will write $f : H(n, q) \longrightarrow \mathbb{R}$ instead of $f : \Sigma_q^n \longrightarrow \mathbb{R}$. Let $f : H(n, q) \longrightarrow \mathbb{R}$. The set $S(f) = \{x \in \Sigma_q^n \mid f(x) \neq 0\}$ is called the *support* of f.

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In [7] Vorob'ev and Krotov proved the following lower bound on the cardinality of the support of an eigenfunction of the Hamming graph:

Theorem 1 (*[7]*). *Let $f : H(n, q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_m and $f \not\equiv 0$. Then*

$$|S(f)| \geq 2^m(q-2)^{n-m} \text{ for } \frac{mq^2}{2n(q-1)} > 2 \text{ and}$$

$$|S(f)| \geq q^n \left(\frac{1}{q-1}\right)^{m/2} \left(\frac{m}{n-m}\right)^{m/2} \left(1-\frac{m}{n}\right)^{n/2} \text{ for } \frac{mq^2}{2n(q-1)} \leq 2.$$

3. Reduction lemma

The set of vertices $x = (x_1, x_2, \dots, x_n)$ of $H(n, q)$ such that $x_i = k$ is denoted by $T_k(i, n)$.

Let $t = (t_1, t_2, \dots, t_n)$ be a vertex of $H(n, q)$. We consider vectors $x = (t_1, \dots, t_{i-1}, k, t_i, \dots, t_n)$ and $y = (t_1, \dots, t_{i-1}, m, t_i, \dots, t_n)$ of length $n + 1$. We note that $x \in T_k(i, n + 1)$ and $y \in T_m(i, n + 1)$, and vector t can be obtained by removing the ith coordinate of vertices x and y. Given a function $f : H(n + 1, q) \longrightarrow \mathbb{R}$, we define the function $f_{n,i,k,m} : H(n, q) \longrightarrow \mathbb{R}$ by the rule $f_{n,i,k,m}(t) = f(x) - f(y)$.

Lemma 1. *Let $f : H(n + 1, q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to an eigenvalue λ. Then $f_{n,i,k,m}(t)$ is an eigenfunction of $H(n, q)$ corresponding to $\lambda + 1$.*

Proof. Let $t = (t_1, t_2, \dots, t_n)$ be a vertex of $H(n, q)$. Consider the vertices $x = (t_1, \dots, t_{i-1}, k, t_i, \dots, t_n)$ and $y = (t_1, \dots, t_{i-1}, m, t_i, \dots, t_n)$ of $H(n + 1, q)$. We note that $x \in T_k(i, n + 1)$ and $y \in T_m(i, n + 1)$. Moreover, x and y are adjacent. Let $x^1, x^2, \dots, x^s$ be the neighbors of x in $T_k(i, n+1)$. We note that y has neighbors $y^1, y^2, \dots, y^s$ in $T_m(i, n+1)$, where x^h and y^h differ only in the ith coordinate. The vector of length n obtained by removing the ith coordinate in x^h (or y^h) is denoted by z^h. Let $p^r = (t_1, \dots, t_{i-1}, r, t_i, \dots, t_n)$ and $P = \{p^0, p^1, \dots, p^{q-1}\}$. Then $N(x) = \{x^1, x^2, \dots, x^s\} \cup \{P \setminus x\}$. Since f is an eigenfunction, we have

$$\lambda f(x) = \sum_{i=1}^{s} f(x^i) + \sum_{i=0}^{q-1} f(p^i) - f(x). \tag{1}$$

Similarly, $N(y) = \{y^1, y^2, \dots, y^s\} \cup \{P \setminus y\}$ and

$$\lambda f(y) = \sum_{i=1}^{s} f(y^i) + \sum_{i=0}^{q-1} f(p^i) - f(y). \tag{2}$$

Subtracting (2) from (1), we find

$$(\lambda + 1)(f(x) - f(y)) = \sum_{i=1}^{s} (f(x^i) - f(y^i)).$$

Then for any vertex t we have $(\lambda + 1)f_{n,i,k,m}(t) = \sum_{j=1}^{s} f_{n,i,k,m}(z^j)$. Since t has the neighbors $z^1, z^2, \dots, z^s$ in $H(n, q)$, we have that $f_{n,i,k,m}(t)$ is an eigenfunction of $H(n, q)$. □

A function $f : H(n + 1, q) \longrightarrow \mathbb{R}$ is called *additive* if for any i, k, m, $1 \leq i \leq n + 1$, $k \in \Sigma_q$ and $m \in \Sigma_q$ the function $f_{n,i,k,m}$ is a constant.

Lemma 2. *Let $f : H(n + 1, q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_1. Then f is an additive function.*

Proof. It is sufficient to prove that for any allowable i, j and p function $f_{n,p,i,j}$ is a constant. Lemma 1 implies that function $f_{n,p,i,j}$ is an eigenfunction of the graph $H(n, q)$ corresponding to λ_0. Since λ_0 has multiplicity 1, any eigenfunction of $H(n, q)$ corresponding to λ_0 is a constant. The lemma is proved. □

4. Additive functions

Lemma 3. *Let $f : H(2, q) \longrightarrow \mathbb{R}$ be an additive function, let $q > 2$, and let $|S(f)| \leq 2(q - 1)$. Then one of the following cases holds:*

1. $f \equiv 0$.
2. *$f(x) = c$ if $x \in T_k(i, 2)$ for some $i \in \{1, 2\}$ and k; $f(x) = 0$ otherwise, where $c \neq 0$ is a constant.*

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