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The coloring game on planar graphs with large girth, by a result on sparse cactuses

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ABSTRACT

We denote by $\chi_g(G)$ the *game chromatic number* of a graph G, which is the smallest number of colors Alice needs to win the *coloring game* on G. We know from Montassier et al. (2012) and, independently, from Wang and Zhang (2011) that planar graphs with girth at least 8 have game chromatic number at most 5.

One can ask if this bound of 5 can be improved for a sufficiently large girth. In this paper, we prove that it cannot. More than that, we prove that there are *cactuses CT* (i.e. graphs in which each edge only belongs to at most one cycle) having $\chi_g(CT) = 5$ despite having arbitrary large girth, and even arbitrary large distance between its cycles.

1. Introduction

In this paper we only consider simple, finite, and undirected graphs. The *length* of a path or cycle is the cardinality of its edge-set. The *girth* $g(G)$ of a graph G is the length of its smallest cycle. A *cactus* is a graph G in which any edge belongs to at most one cycle. The *cycle-distance* of a cactus is the length of its smallest path between two vertices belonging to different cycles. For a vertex v, we call v-*leaf* a vertex of degree 1 (or *leaf*) whose neighbor is v.

The *coloring game* on a graph G is a two-player non-cooperative game on the vertices of G, introduced by Brams [8] and rediscovered ten years after by Bodlaender [3]. Given a set of k colors, Alice and Bob take turns coloring properly an uncolored vertex, with Alice playing first. Therefore, Alice and Bob are constructing a proper partial coloring of G of increasing number of colored vertices. The game ends in Alice's victory when the graph is fully colored, of in Bob's victory if an uncolored vertex cannot be properly colored. The *game chromatic number* $\chi_g(G)$ of G is the smallest number of colors ensuring Alice's victory. This graph invariant has been extensively studied in the past twenty years, see for example [9,12,16,17].

In [3], Bodlaender proved that every forest F has $\chi_g(F) \leq 5$, and exhibited trees T with $\chi_g(T) \geq 4$. In [7], Faigle et al. showed that every forest F has $\chi_g(F) \leq 4$. Conditions for trees to have game chromatic number 3 were recently studied by Dunn et al. [6].

A graph is said to be $(1, k)$-*decomposable* if its edge set can be partitioned into two sets, one inducing a forest and the other inducing a graph with maximum degree at most k. Using the notion of *marking game* introduced by Zhu in [15], He et al. [10] observed that every $(1, k)$-decomposable graph has $\chi_g(G) \leq k + 4$, then deduced upper bounds for the game chromatic number of planar graphs with given girth. Among other results, they proved that planar graphs with girth at least 11 are $(1, 1)$-decomposable, and therefore their game chromatic number is at most 5. Later, were proved successively to be $(1, 1)$-decomposable: planar graphs with girth 10 by Bassa et al. [2], girth 9 by Borodin et al. [5], and girth 8 by Montassier et al. [11] and Wang and Zhang [14] independently. There exist planar graphs with girth 7 that are not $(1, 1)$-decomposable.

Borodin et al. [4] gave conditions for planar graphs with no small cycles except triangles to be $(1, 1)$-decomposable, in terms of distance between the triangles and of minimal length of a non-triangle cycle. In [13], Sidorowicz, arguing that

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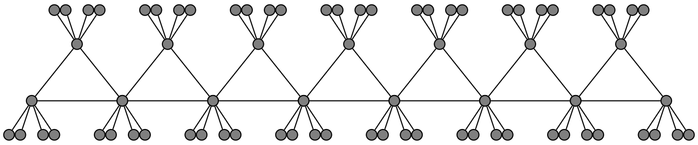

Fig. 1. A cactus with game chromatic number 5 [13].

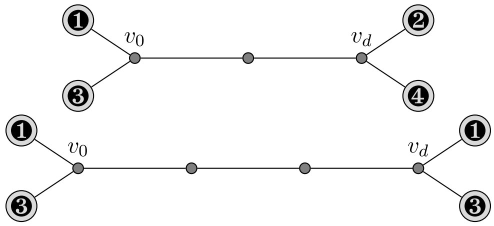

Fig. 2. Two winning paths (Definition 2).

cactuses are $(1, 1)$-decomposable, showed that every cactus CT has $\chi_g(CT) \leq 5$. Moreover, she exhibited a cactus with game chromatic number 5, depicted in Fig. 1. As one can see, this cactus has intersecting triangles.

Initially, our work was a try to answer the following question:

Question 1. *Is there an integer g such that every planar graph G with girth at least g has* $\chi_g(G) \leq 4$*?*

We answer negatively, with a result going way beyond the question we initially asked.

Theorem 1. *For any integers d, k, there are cactuses CT with girth at least k, cycle-distance at least d and* $\chi_g(G) = 5$.

This proves that the upper bound of 5 for the game chromatic number of the classes of $(1, 1)$-decomposable graphs considered in [2,4,5,10,11,14] are best possible.

2. Proof of Theorem 1

We consider Alice and Bob playing the coloring game on a graph G with a set of four colors $\mathcal{C} = \{❶, ❷, ❸, ❹\}$. At each time of the game, we denote by $\phi(v)$ the color of a vertex v (if v is colored), and by $\Phi(v)$ the set of colors in the neighborhood of v: if v is uncolored, then this is the set of colors forbidden for v. An uncolored vertex v with $\Phi(v) = \mathcal{C}$ is called *surrounded*. Bob wins if he can surround a vertex. For any vertex v, a v*-leaf* is a leaf whose neighbor is v.

We give further constructions of G. For now just assume that *every cycle of G is odd* and that *every non-leaf vertex of G is adjacent to a large number of leaves*, say at least 8 leaves.

We also give further Bob's strategy in details, but assume that *Bob only plays on the leaves of G*. Also, for any vertex v, when Bob colors a v-leaf, he uses a color that is not already in $\Phi(v)$. If Alice colors a v-leaf during the game, then Bob always colors another v-leaf if possible. So, during the game, we can assume that *for any uncolored vertex v, there is always at least one uncolored leaf of v*, since the opposite would imply v is surrounded. Moreover, coloring a v-leaf for Alice does not suppress any possibility for Bob (this forbid Bob to color v with the color Alice used, but Bob had no intention to color v anyway) and only increases $\Phi(v)$, coloring a leaf is always unoptimal for Alice with regard to Bob's selected strategy. So we assume *Alice never colors a leaf during the game*.

We describe some *winning positions* for Bob (i.e. partial colorings from where Bob has a winning strategy) in the following lemmas. In the description of every winning position, we assume that it is Alice's turn to play.

Definition 2. Suppose that G contains a path of length d, $P = v_0 \dots v_d$, of uncolored non-leaf vertices. Also suppose $|\Phi(v_0)| \geq 2$, say $\{❶, ❸\} \subseteq \Phi(v_0)$. We say that P is a *winning path* if (see Fig. 2):

- d is odd and $\{❶, ❸\} \subseteq \Phi(v_d)$.
- d is even and $\{❷, ❹\} \subseteq \Phi(v_d)$.

Lemma 3. *Graph G is in a winning position for Bob if it contains a winning path.*

Proof. Recall that, by assumption, every vertex of P has at least one uncolored leaf.

In the case $d = 0$, P is reduced to a single surrounded vertex and Bob wins.

The case $d = 1$ corresponds to an edge v_0v_1 with v_0 and v_1 uncolored and $|\Phi(v_0) \cap \Phi(v_1)| \geq 2$, say $\{❶, ❸\} \subseteq \Phi(v_0) \cap \Phi(v_1)$. If Alice colors v_0, she has to use an even color, Bob colors a v_1-leaf with the other even color, surrounds v_1, and wins. By

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