ARTICLE IN PRESS

Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A note on the surviving rate of 1-planar graphs☆

Jiangxu Kong [a,b], Lianzhu Zhang [b,*]

[a] Department of Mathematics, China Jiliang University, Zhejiang 310018, China
[b] School of Mathematical Science, Xiamen University, Fujian 361005, China

ARTICLE INFO

Article history:
Received 29 January 2015
Received in revised form 25 September 2016
Accepted 2 November 2016
Available online xxxx

Keywords:
Firefighter problem
Surviving rate
1-planar graph

ABSTRACT

For a connected graph G, suppose that a fire breaks out at its vertex and a firefighter starts to protect vertices. At each time interval, the firefighter protects k vertices not yet on fire. At the end of each time interval, the fire spreads to all the unprotected vertices that have a neighbor on fire. The k-surviving rate $\rho_k(G)$ of G is defined to be the expected percentage of vertices saved if the fire breaks out at a random vertex. In this note, we consider the surviving rate of 1-planar graphs, and show that every 1-planar graph G has $\rho_6(G) > \frac{1}{163}$.

1. Introduction

The firefighter problem was introduced by Hartnell [7] in 1995 at the 25th Manitoba Conference on Combinatorial Mathematics and Computing. Assume that a fire breaks out at a vertex v of G. A firefighter chooses a vertex not yet on fire to protect. Then the firefighter and the fire alternately move on the graph. Once a vertex has been chosen by the firefighter, it is considered protected or safe from any further moves of the fire. After the firefighter's move, the fire makes its move by spreading to all vertices which are adjacent to the vertices on fire, except for those that are protected. The process stops when the fire can no longer spread. The objective of the firefighter is to save the maximum number of vertices, i.e., the number of vertices which are not burning when the process ends.

The problem can be considered as a simplified deterministic model of the spread of fire, diseases, and computer viruses. And it may also be viewed as a one-person game, with only the firefighter using strategy.

Let $\mathrm{sn}(v)$ denote the maximum number of vertices in G that the firefighter can save when a fire breaks out at vertex v. Determining for a graph G, vertex $v \in V(G)$ and an integer l, whether $\mathrm{sn}(v) \geq l$ is NP-complete, even when G is restricted to cubic graphs [8], bipartite graphs [11] and trees with maximum degree three [4]. For a survey of related results the reader is referred to [5].

The *surviving rate* $\rho(G)$ of a graph G of order n, was introduced by Cai and Wang [2], and is defined to be the expected percentage of vertices that can be saved when a fire breaks out at one vertex of the graph. More generally, for an integer $t \geq 1$, the k-firefighter problem is the same as the firefighter problem, except that at each move, the firefighter protects k vertices. We use $\mathrm{sn}_k(v)$ to denote the maximum number of vertices in G that the firefighter can save when a fire breaks out at vertex v. The *k-surviving rate* $\rho_k(G)$ of a graph G with n vertices is defined by

$$\rho_k(G) = \frac{\sum_{v\in V(G)} \mathrm{sn}_k(v)}{n^2}.$$

☆ This work is supported by NSFC (Nos. 11471273, 11671186) and ZJNSFC (No. LQ17A010005).
* Corresponding author.
E-mail addresses: kjx@cjlu.edu.cn (J. Kong), zhanglz@xmu.edu.cn (L. Zhang).

http://dx.doi.org/10.1016/j.disc.2016.11.005

In particular, $\rho_1(G) = \rho(G)$. For example, for paths P_n with n vertices, $\rho(P_n) = 1 - \frac{2}{n} + \frac{2}{n^2}$. And since $\text{sn}(v) = 1$ for all vertices $v \in V(K_n)$, we have that $\rho(K_n) = \frac{1}{n}$. Thus for all graphs G of order n, $\frac{1}{n} \leq \rho(G) < 1$.

Wang et al. [14] proved that for any $k \geq 1$, the k-surviving rate of almost all graphs is arbitrarily close to zero and therefore they began studying classes of special graphs, e.g., planar graphs, with the k-surviving rate bounded away from zero. In [3], Esperet et al. defined the firefighter number for a class of graphs $\mathcal{C}$. Formally, the firefighter number for a class of graphs $\mathcal{C}$, denoted by $ff(\mathcal{C})$, is the minimum integer k such that there exist an $\epsilon > 0$ and an integer N so that every $G \in \mathcal{C}$ with at least N vertices has $\rho_k(G) > \epsilon$. Since $\rho(K_{2,n}) = \frac{2}{n+2}$, $ff(\mathcal{P}) \geq 2$, where $\mathcal{P}$ is the class of planar graphs. The firefighter number of the class of planar graphs with girth at least seven is one [15]. The firefighter number of planar graphs with girth five and six remains open. Two independent proofs have appeared that $ff(\mathcal{P}) \leq 4$ [3,9], this was recently improved to $ff(\mathcal{P}) \leq 3$ [6,10] and it was conjectured that $ff(\mathcal{P}) = 2$ [3]. This conjecture was confirmed for planar graphs without 3-cycles [3], without 4-cycles [16], without 5-cycles [18], or without 6-cycles [13]. Other results on the surviving rate of graphs are referred to [1] and [17].

A graph that can be drawn in the Euclidean plane so that every edge is crossed by at most k others is called a *k-planar graph*, we use $\mathcal{P}_k$ to denote the class of k-planar graphs. Obviously, $\mathcal{P}_0$ are the class of planar graphs and $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \cdots$. In this note, we will investigate the surviving rate of 1-planar graphs.

Theorem 1 (*[14]*). *Let $l \geq 2$ be an integer. Let G be a graph with n vertices and m edges such that $n \geq 2l$ and $m \leq ln$. Then $\rho_{2l-1}(G) \geq \frac{2}{5l}$.*

Theorem 2 (*[12]*). *Let G be a k-planar graph with n vertices and m edges. Then*

$$m \leq \begin{cases} (k+3)(n-2), & \text{if } 0 \leq k \leq 4; \\ 4.108\sqrt{k}n, & \text{otherwise.} \end{cases}$$

Theorems 1 and 2 imply that for every 1-graph G has $\rho_7(G) > \frac{1}{10}$, thus, $ff(\mathcal{P}_1) \leq 7$. In this note, we obtain the following result which implies that $ff(\mathcal{P}_1) \leq 6$.

Theorem 3. *Let G be a 1-planar graph. Then $\rho_6(G) > \frac{1}{163}$.*

2. Proof of Theorem 3

A *1-plane* graph which is a particular drawing of a 1-planar graph in the Euclidean plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. For a 1-plane graph G, each pair x_1y_1, x_2y_2 of edges that cross each other at a crossing point z, their end vertices are pairwise distinct. Let $C(G)$ be the set of all crossing points and $E^*(G)$ the non-crossed edges in G. An *associated plane graph* $G^\times = (V(G^\times), E(G^\times))$ of G is the plane graph such that $V(G^\times) = V(G) \cup C(G)$ and $E(G^\times) = E^*(G) \cup \{xz, yz | xy \in E(G) \setminus E^*(G)$ and z is the crossing point on $xy\}$. Thus, each of the crossing point becomes an actual vertex of degree four in $G^\times$. For convenience, we still call the new vertices in $G^\times$ crossing vertices, and the edge in $E(G)$ which contains a crossing vertex is called a *crossing edge*. Denote $V(G^\times)$, $E(G^\times)$ and $F(G^\times)$ the sets of vertices, edges and faces of $G^\times$, respectively. And denote $V_k(G)$, $V_k^-(G)$ and $V_k^+(G)$ the sets of vertices in G with degree k, at most k and at least k, respectively. We say a vertex v is k-vertex (k^--vertex or k^+-vertex) if $v \in V_k$ ($v \in V_k^-(G)$ or $v \in V_k^+(G)$). It is clear that $d_G(v) = d_{G^\times}(v)$ for every vertex $v \in V(G)$.

In order to show the main result, we first prove the following useful lemma.

Lemma 1. *Assume that G is a 1-plane graph with order n and $G^\times$ is its associated plane graph. Let w be a weight function on $F(G^\times) \cup V(G^\times)$ such that*

$$w(x) \geq \begin{cases} 0, & x \in F(G^\times) \cup C(G^\times); \\ \alpha, & x \in V(G), sn_k(x) \geq n - l; \\ \beta, & \text{otherwise}, \end{cases}$$

where $\alpha \leq 0 \leq \beta$, $l + k \leq n$ and $\beta l \leq -\alpha k$. If the total weight is negative, then $\rho_k(G) > \frac{\beta}{\beta - \alpha}$.

Proof. Let $V^g = \{x \mid x \in V(G), sn_k(x) \geq n - l\}$ and $n^g = |V^g|$. According to the assumption, it follows that

$$\begin{aligned} 0 &> \sum_{v \in V(G^\times)} w(v) + \sum_{f \in F(G^\times)} w(f) \\ &= \sum_{v \in C(G)} w(v) + \sum_{v \in V^g} w(v) + \sum_{v \in V(G) \setminus V^g} w(v) + \sum_{f \in F(G^\times)} w(f) \\ &\geq \sum_{v \in V^g} w(v) + \sum_{v \in V(G) \setminus V^g} w(v) \\ &\geq \alpha n^g + \beta(n - n^g). \end{aligned}$$

This gives $n^g > \frac{\beta}{\beta - \alpha} n$.