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Note

On the critical group of the missing Moore graph

Joshua E. Ducey

Department of Mathematics and Statistics, James Madison University, Harrisonburg, VA 22807, United States

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ABSTRACT

We consider the critical group of a hypothetical Moore graph of diameter 2 and valency 57. Determining this group is equivalent to finding the Smith normal form of the Laplacian matrix of such a graph. We show that all of the Sylow p-subgroups of the critical group must be elementary abelian with the exception of $p = 5$. We prove that the 5-rank of the Laplacian matrix determines the critical group up to two possibilities.

1. Introduction

Consider a simple graph with diameter d and girth $2d + 1$. Such a graph is necessarily regular, and is known as a *Moore graph*. Another characterization: Moore graphs are the regular graphs of diameter d and valency k that achieve the upper bound on number of vertices

$$1 + \sum_{i=1}^{d} k \cdot (k-1)^{i-1}.$$

We will denote such a graph of diameter d and valency k as a Moore(k, d). It was shown in [8] that for Moore graphs of diameter 2, one must have the valency $k \in \{2, 3, 7, 57\}$. The 5-cycle, the Petersen graph, and the Hoffman–Singleton graph are the unique graphs satisfying the first three respective degrees. Neither the existence nor uniqueness of a Moore graph of diameter 2 and valency 57 have yet been established.

There has been some work on determining algebraic properties of such a graph, especially regarding its automorphism group [2,10]. It is known that a Moore$(57, 2)$ possesses very few automorphisms, if any at all. For a more recent result on the enumeration of independent sets in such a graph, see [1].

In this paper we investigate the structure of the *critical group* of a Moore$(57, 2)$. We define this abelian group formally in the next section, but we mention here that it is an important graph invariant that has been widely studied and goes by many names in the literature (sandpile group, Jacobian group, Picard group). The group comes from the Laplacian matrix of the graph and has order equal to the number of spanning trees of the graph. The critical group can also be understood in terms of a certain "chip-firing" game on the vertices of the graph [3], [7].

E-mail address: duceyje@jmu.edu.

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In Section 2 we give formal definitions, and describe the relation between the critical group and the Laplacian matrix of a graph. Our main results are Theorems 3.1 and 3.2, which together show that the 5-rank of the Laplacian matrix of a Moore(57, 2) determines the critical group to within two possibilities. We state these theorems immediately below for the interested reader. They will be proved in Section 3. The critical group of a graph Γ is denoted $K(\Gamma)$. Let $Syl_p(K(\Gamma))$ denote the Sylow p-subgroup of the critical group.

Theorem 3.1. *Let Γ denote a Moore(57, 2) graph. Then for some nonnegative integers e_1, e_2, e_3 we have*

$$K(\Gamma) \cong (\mathbb{Z}/2\mathbb{Z})^{1728} \oplus (\mathbb{Z}/13\mathbb{Z})^{1519} \oplus (\mathbb{Z}/5\mathbb{Z})^{e_1} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{e_2} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_3}.$$

Theorem 3.2. *Let Γ be a Moore(57, 2) graph. Let e_0 denote the rank of the Laplacian matrix of Γ over a field of characteristic 5. Then either*

$$Syl_5(K(\Gamma)) \cong (\mathbb{Z}/5\mathbb{Z})^{1520-e_0} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{1732-e_0} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_0-3}$$

or

$$Syl_5(K(\Gamma)) \cong (\mathbb{Z}/5\mathbb{Z})^{1521-e_0} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{1730-e_0} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_0-2}.$$

2. Preliminaries

Let Γ be a simple graph with some fixed ordering of the vertex set $V(\Gamma)$. Then the *adjacency matrix* of Γ is a square matrix $A = (a_{i,j})$ with rows and columns indexed by $V(\Gamma)$, where

$$a_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ is adjacent to vertex } j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $D = (d_{i,j})$ be a matrix of the same dimensions as A with

$$d_{i,j} = \begin{cases} \text{the degree of vertex } i, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, set $L = D - A$. The matrix L is called the *Laplacian matrix* of the graph Γ, and will be our primary focus.

Let $\mathbb{Z}^{V(\Gamma)}$ denote the free abelian group on the vertex set of Γ. Then the Laplacian L can be understood as describing a homomorphism:

$$L : \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}.$$

We will usually use the same symbol for both the matrix and the map. The cokernel of L,

$$\operatorname{coker} L = \mathbb{Z}^{V(\Gamma)}/\operatorname{Im}(L),$$

always has free rank equal to the number of connected components of Γ. The torsion subgroup of $\operatorname{coker} L$ is known as the *critical group* of Γ, and is denoted $K(\Gamma)$. It is an interesting fact that for a connected graph Γ, the order of $K(\Gamma)$ is equal to the number of spanning trees of Γ. See [3] or [9] for proofs of these basic facts and more information. One way to compute the critical group of a graph is by finding the *Smith normal form* of L.

Recall that if M is any $m \times n$ integer matrix then one can find square, unimodular (i.e., unit determinant) matrices P and Q so that $PMQ = S$, where the matrix $S = (s_{i,j})$ satisfies:

(1) $s_{i,i}$ divides $s_{i+1,i+1}$ for $1 \le i < \min\{m, n\}$
(2) $s_{i,j} = 0$ for $i \neq j$.

Then S is known as the Smith normal form of M, and it is not hard to see that

$$\operatorname{coker} M \cong \mathbb{Z}/s_{1,1}\mathbb{Z} \oplus \mathbb{Z}/s_{2,2}\mathbb{Z} \oplus \cdots.$$

This particular decomposition of $\operatorname{coker} M$ is the invariant factor decomposition, and the integers $s_{i,i}$ are known as the *invariant factors* of M. The prime power factors of the invariant factors of M are known as the *elementary divisors* of M.

The concept of Smith normal form generalizes nicely when one replaces the integers with any principal ideal domain (PID), as is well known (see, for example, [6]). In what follows J and I will be used to denote the all-ones matrix and the identity matrix, respectively, of the correct sizes.

3. The critical group of a Moore(57, 2)

Throughout the rest of the paper we let Γ denote a Moore(57, 2) graph. It follows easily from the definitions that Γ is strongly regular with parameters

$$v = 3250, k = 57, \lambda = 0, \mu = 1$$