Discrete Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

# Packing chromatic number under local changes in a graph

Boštjan Brešar [a,b], Sandi Klavžar [a,b,c,*], Douglas F. Rall [d], Kirsti Wash [e]

[a] Faculty of Natural Sciences and Mathematics, University of Maribor, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia
[c] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia
[d] Department of Mathematics, Furman University, Greenville, SC, USA
[e] Department of Mathematics, Trinity College, Hartford, CT, USA

ARTICLE INFO

*Article history:*
Received 23 May 2016
Received in revised form 20 September 2016
Accepted 24 September 2016
Available online xxxx

*Keywords:*
Packing chromatic number
Cubic graph
Subdivision
Contraction

ABSTRACT

The packing chromatic number $\chi_\rho(G)$ of a graph $G$ is the smallest integer $k$ such that there exists a $k$-vertex coloring of $G$ in which any two vertices receiving color $i$ are at distance at least $i+1$. It is proved that in the class of subcubic graphs the packing chromatic number is bigger than 13, thus answering an open problem from Gastineau and Togni (2016). In addition, the packing chromatic number is investigated with respect to several local operations. In particular, if $S_e(G)$ is the graph obtained from a graph $G$ by subdividing its edge $e$, then $\lfloor \chi_\rho(G)/2 \rfloor + 1 \le \chi_\rho(S_e(G)) \le \chi_\rho(G) + 1$.



## 1. Introduction

Many variations of the classical graph coloring have been introduced, several of which involve graph distance, which as a condition is usually imposed on the vertices that are given the same color. In this paper we study packing colorings defined as follows. The *packing chromatic number* $\chi_\rho(G)$ of $G$ is the smallest integer $k$ such that $V(G)$ can be partitioned into subsets $X_1, \ldots, X_k$, where $X_i$ induces an $i$-packing; that is, vertices of $X_i$ are pairwise at distance more than $i$. Equivalently, a *$k$-packing coloring* of $G$ is a function $c : V(G) \to [k]$, where $[k] = \{1, \ldots, k\}$, such that if $c(u) = c(v) = i$, then $d_G(u, v) > i$, where $d_G(u, v)$ is the usual shortest-path distance between $u$ and $v$ in $G$.

The concept of the packing chromatic number was introduced in [10] and given the name in [3]. The problem intuitively appears more difficult than the standard coloring problem. Indeed, the packing chromatic number is intrinsically more difficult due to the fact that determining $\chi_\rho$ is NP-complete even when restricted to trees [7]. On the other hand, Argiroffo et al. discovered that the packing coloring problem is solvable in polynomial time for several nontrivial classes of graphs [2]. In addition, the packing chromatic number was studied on hypercubes [10,16], Cartesian product graphs [11,13], and distance graphs [6,15].

In the seminal paper [10] the following problem was posed: does there exist an absolute constant $M$, such that $\chi_\rho(G) \le M$ holds for any subcubic graph $G$. (Recall that a graph is *subcubic*, if its largest degree is bounded by 3.) This problem led to a lot of research but remains unsolved at the present. In particular, the packing chromatic number of the infinite hexagonal lattice is 7 (the upper bound being established in [8], the lower bound in [12]), hence the packing chromatic number of any subgraph of the hexagonal lattice is bounded by 7. The same bound also holds for subcubic trees as follows from a result of Sloper [14]. For the (subcubic) family of base-3 Sierpiński graphs the packing chromatic number was bounded by 9 in [4].

* Corresponding author at: Faculty of Mathematics and Physics, University of Ljubljana, Slovenia.
*E-mail address:* sandi.klavzar@fmf.uni-lj.si (S. Klavžar).

http://dx.doi.org/10.1016/j.disc.2016.09.030
0012-365X/

The exact value of the packing chromatic number of some additional subcubic graphs was determined in [5]. Very recently, Gastineau and Togni [9] found a cubic graph with packing chromatic number equal to 13 and posed an open problem which intrigued us: does there exist a cubic graph with packing chromatic number larger than 13?

We proceed as follows. In Section 2 we prove that the answer to the above question is positive. More precisely, we construct a cubic graph on 78 vertices with packing chromatic number at least 14. Since a key technique in the related proof is edge subdivision, we give a closer look at this operation with respect to its effect on the packing chromatic number. In particular, the packing chromatic number does not increase by more than 1 when an edge of a graph is subdivided, but can decrease by at least 2. In addition, we prove that the lower bound for the packing chromatic number of an edge-subdivided graph is greater than half of the packing chromatic number of the original graph. Then, in Section 3, we investigate the effect of the following local operations on the packing chromatic number: a vertex deletion, an edge deletion, and an edge contraction. In particular, we demonstrate that the difference $\chi_\rho(G) - \chi_\rho(G-e)$ can be arbitrarily large.

## 2. Edge subdivision

In this section we consider the packing chromatic number with respect to the edge-subdivision operation. If $e$ is an edge of a graph $G$, then let $S_e(G)$ denote the graph obtained from $G$ by subdividing the edge $e$ with precisely one new vertex. The graph obtained from $G$ by subdividing all its edges is denoted $S(G)$.

The following theorem is the key for the answer of the above-mentioned question of Gastineau and Togni.

**Theorem 2.1.** *Suppose that there exists a constant $M$ such that $\chi_\rho(H) \le M$ holds for any subcubic graph $H$. If $G$ is a subcubic graph such that $\chi_\rho(G) = M$, then either $\chi_\rho(S_e(G)) \le M-2$ for any $e \in E(G)$, or* $\operatorname{diam}(G) \ge \lceil \frac{M}{2} \rceil - 2$.

**Proof.** Let $G$ be a subcubic graph such that $\chi_\rho(G) = M$, where $\chi_\rho(H) \le M$ for every subcubic graph $H$. If $\chi_\rho(S_e(G)) \le M-2$ holds for any $e \in E(G)$, there is nothing to be proved. Hence assume that there exists an edge $e \in E(G)$ such that $\chi_\rho(S_e(G)) \ge M-1$. Let $G'$ be the graph obtained from $G$ by subdividing the edge $e$, and let $x'$ be the new vertex. Let $G''$ be a copy of $G'$, with $x''$ playing the role of $x'$. Let now $\widehat{G}$ be the graph obtained from the disjoint union of $G'$ and $G''$ by connecting $x'$ with $x''$.

Note first that $\widehat{G}$ is a subcubic graph, and hence by the theorem's assumption, $\chi_\rho(\widehat{G}) \le M$. Let $c$ be an arbitrary optimal packing coloring of $\widehat{G}$. Because $c$ restricted to $G'$ (resp. $G''$) is a packing coloring of $G' = S_e(G)$ (resp. $G''$), $c$ uses at least $M-1$ colors. We claim that $\operatorname{diam}(\widehat{G}) \ge M-1$. If $c$ colors a vertex $u'$ of $G'$ and a vertex $u''$ of $G''$ by the color $M$, then $d_{\widehat{G}}(u', u'') > M$, and the claim follows. Otherwise, we assume that $c$ restricted to $G'$ does not use the color $M$. If also $G''$ does not use color $M$, then since $\chi_\rho(G') \ge M-1$ and $\chi_\rho(G'') \ge M-1$, there exist vertices $v'$, $v''$ in $G'$, resp. $G''$, with $c(v') = c(v'') = M-1$, and consequently, $\operatorname{diam}(\widehat{G}) > M-1$ as desired. So assume that color $M$ is present on $G''$ (and not on $G'$). Color $M-1$ must be present on $G'$, for otherwise $\chi_\rho(G') \le M-2$. If color $M-1$ is also used on $G''$, then it again follows that $\operatorname{diam}(\widehat{G}) > M-1$. Hence we are left with the situation that color $M$ is present on $G''$ and not on $G'$, while $M-1$ is used on $G'$ and not on $G''$. We now claim that the color $M-2$ is present in both $G'$ and $G''$. For if this is not the case, then in any of $G'$ or $G''$ that is missing color $M-2$ relabeling all vertices colored with the highest color by the color $M-2$ would yield an $(M-2)$-packing coloring of $G'$ or $G''$, which is again not possible. If $w'$, $w''$ are the vertices in $G'$, resp. $G''$, with $c(w') = c(w'') = M-2$, then $d_{\widehat{G}}(w', w'') \ge M-1$. This in turn implies $\operatorname{diam}(\widehat{G}) \ge M-1$, and so the claim is proved.

Consider again vertices $w'$, $w''$ in $G'$, resp. $G''$, with $c(w') = c(w'') \ge M-2$. Since $\operatorname{diam}(G') \ge d_{\widehat{G}}(w', x')$ and $\operatorname{diam}(G'') \ge d_{\widehat{G}}(w'', x'')$, we infer that

$$\begin{aligned} 2\operatorname{diam}(G') + 1 &= \operatorname{diam}(G') + \operatorname{diam}(G'') + 1 \\ &\ge d_{\widehat{G}}(w', x') + d_{\widehat{G}}(w'', x'') + 1 \\ &= d_{\widehat{G}}(w', w'') \\ &\ge M - 1. \end{aligned}$$

Hence $\operatorname{diam}(G') \ge \lceil \frac{M}{2} \rceil - 1$. Since clearly $\operatorname{diam}(G') \le \operatorname{diam}(G) + 1$ holds, we conclude that

$$\operatorname{diam}(G) \ge \operatorname{diam}(G') - 1 \ge \left\lceil \frac{M}{2} \right\rceil - 2. \quad \square$$

**Corollary 2.2.** *There exists a cubic graph with packing chromatic number larger than* 13.

**Proof.** Let $G_{38}$ be the cubic graph of order 38 with diameter 4 from [1] shown in Fig. 1.

From [9, Proposition 6] we know that $\chi_\rho(G_{38}) = 13$. We have checked by computer that $\chi_\rho(S_e(G_{38})) = 12$ holds for any edge $e$ of $G_{38}$. Assuming that $M = 13$ is the constant of Theorem 2.1, this theorem implies that $\operatorname{diam}(G_{38}) \ge \lceil \frac{13}{2} \rceil - 2 = 5$. However, since the diameter of $G_{38}$ equals 4, we infer that $M$ cannot be 13. $\square$

A closer look to the proof of Theorem 2.1 reveals that the graph constructed from two copies $G'_{38}$ and $G''_{38}$ of edge-subdivided $G_{38}$ by connecting the vertices $x'$ and $x''$ is a graph of order 78, say $G_{78}$ schematically shown in Fig. 2, such that $\chi_\rho(G_{78}) \ge 14$.