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A large family of cospectral Cayley graphs over dihedral groups

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ABSTRACT

The adjacency spectrum of a graph Γ, which is denoted by $Spec(\Gamma)$, is the multiset of eigenvalues of its adjacency matrix. We say that two graphs Γ and Γ' are cospectral if $Spec(\Gamma) = Spec(\Gamma')$. In this paper for each prime number p, $p \geq 23$, we construct a large family of cospectral non-isomorphic Cayley graphs over the dihedral group of order $2p$.
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1. Introduction

Let G be a finite group and S be an inverse closed subset, $S = S^{-1} = \{s^{-1} : s \in S\}$, of the set $G \setminus \{e\}$, where e denotes the identity element of the group G. The Cayley graph $Cay(G, S)$ is the graph whose vertex set is G and two vertices $a, b \in G$ are adjacent whenever $ab^{-1} \in S$. The adjacency spectrum of graph Γ, which is denoted by $Spec(\Gamma)$, is the multiset of eigenvalues of its adjacency matrix. Two graphs Γ and Γ' are called cospectral if $Spec(\Gamma) = Spec(\Gamma')$, and for these two cospectral graphs, we say Γ' is a cospectral mate for Γ. Finding large families of cospectral graphs with special properties is a difficult task and have been investigated in numerous articles [2,5–12]. For more details about cospectral graphs and related topics one can see [1,3,13] and references therein. In [12], by using the Seidel switching, a cospectral family $\mathcal{F}_n$ of size greater than $\frac{2^{n/6}}{80}$ of 8-regular simple graphs on n points, for $n > 8$, has been constructed. By some general techniques such as NEPS product or special types of switching (for example GM-switching), large families of cospectral graphs have been constructed [6,10,11]. In contrast, we know of only the papers [2,5,9] in which cospectral Cayley graphs are presented. One of the most interesting results is by Babai [5], who showed that, for each integer number k, $k \geq 2$, and each prime number p, $p > 64k$, there are k pairwise non-isomorphic cospectral Cayley graphs over the dihedral group D_{2p}. Also, in [9], the authors constructed some non-isomorphic cospectral Cayley graphs over the group $PSL_d(\mathbb{F}_q)$, for some special values of d and q. Recently, in [2], cospectral Cayley graphs over finite groups are studied and infinite families of non-isomorphic Cayley graphs over finite groups are constructed.

In [2], for each prime number $p \geq 13$, the authors gave two non-isomorphic 6-regular cospectral Cayley graphs over the dihedral group D_{2p}.

In this paper, by generalizing the construction method of cospectral Cayley graphs over dihedral groups which is introduced in [2], we construct a large family of non-isomorphic cospectral Cayley graphs over the dihedral group D_{2p}, $p \geq 23$ prime. Our construction shows that the total number of non-isomorphic cospectral Cayley graphs over the dihedral group D_{2p} is exponential in terms of p.

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Our main results are as follows:

Theorem 1.1. *Let $p \geq 23$ be a prime number. Then for each integer number d, $6 \leq d \leq 2p-7$, there exist at least two non-isomorphic cospectral d-regular Cayley graphs over the dihedral group D_{2p}.*

Theorem 1.2. *Let $p \geq 23$ be a prime number, d be a positive integer such that $6 \leq d \leq p+6$ and $\overline{d} = 2p - d - 1$. Then there exist $C(\frac{p-1}{2}, \lfloor \frac{d}{2} \rfloor - 3)$ pairwise non-isomorphic cospectral d-regular ($\overline{d}$-regular) Cayley graphs over the dihedral group D_{2p}.*

2. Preliminaries

In the following, we state some definitions and results which are needed in the sequel. For more details one can see [2,5].

Let $p \geq 13$ be a prime number and $D_{2p} = \langle \sigma, \tau \mid \sigma^2 = \tau^p = 1, (\sigma\tau)^2 = 1 \rangle$ denotes the dihedral group of order $2p$. Any inverse closed subset S of $D_{2p} \setminus \{e\}$, can be written as follows:

$$S = \{\tau^{k_1}, \ldots, \tau^{k_a}\} \cup \{\tau^{l_1}\sigma, \ldots, \tau^{l_b}\sigma\} \quad (1)$$

for some uniquely determined integers $k_1, \ldots, k_a$, $0 < k_1 < \cdots < k_a \leq p-1$, and $l_1, \ldots, l_b$, $0 \leq l_1 < \cdots < l_b \leq p-1$; the latter holds: for p is odd and so every element of order 2 in D_{2p} has a unique form as $\tau^i\sigma$ for some unique integer $i \in \{0, \ldots, p-1\}$.

Remark 2.1. By the above notation (1), we denote $S_\sigma := \{l_1, \ldots, l_b\}$ and $S_\tau := \{k_1, \ldots, k_a\}$. Note that integers in S_σ correspond to elements of order 2 in S.

Theorem 2.2 (*Corollary 4.2 of [5]*). *Let n be an odd integer and for each integer number c, $0 \leq c \leq n-1$, $\beta(c)$ denotes the number of solutions of the congruence*

$$x - y \equiv c \mod n, \quad x, y \in S_\sigma = \{l_1, \ldots, l_b\}.$$

Then, the set $S_\tau = \{k_1, \ldots, k_a\}$ and the function β determine the spectrum of the Cayley graph $\Gamma_S = Cay(D_{2n}, S)$.

The automorphism groups of the dihedral groups are easy to describe:

Theorem 2.3. *Suppose $n > 2$ is an integer and D_{2n} is the dihedral group of order $2n$. Then the automorphism group of D_{2n}, which is denoted by $Aut(D_{2n})$, is isomorphic to $\mathbb{Z}_n^\times \ltimes \mathbb{Z}_n$ and for an arbitrary automorphism of D_{2n} such as $\alpha_{s,t}$, $s \in \mathbb{Z}_n^\times$ and $t \in \mathbb{Z}_n$, we have $\alpha_{s,t}(\tau^i\sigma) = \tau^{is+t}\sigma$ and $\alpha_{s,t}(\tau^i) = \tau^{is}$.*

The dihedral group D_{2p}, p a prime number, is a CI-group [4], which means:

Theorem 2.4 (*Theorem 5.1 of [5]*). *The two Cayley graphs $\Gamma_S = Cay(D_{2p}, S)$ and $\Gamma_T = Cay(D_{2p}, T)$ of D_{2p} (p prime) are isomorphic if and only if there is an automorphism α of D_{2p} which maps S onto T.*

3. Proofs of main results

In this section, first we give some lemmas which are needed to prove our main results.

Theorem 3.1 (*Theorem 1.3 of [2]*). *Let $p \geq 13$ be a prime number and D_{2p} be the dihedral group of order $2p$. Suppose $S = \{\sigma, \tau\sigma, \tau^2\sigma, \tau^6\sigma, \tau^8\sigma, \tau^{11}\sigma\}$ and $T = \{\sigma, \tau^2\sigma, \tau^4\sigma, \tau^5\sigma, \tau^{10}\sigma, \tau^{11}\sigma\}$. Then the Cayley graphs $\Gamma_S = Cay(D_{2p}, S)$ and $\Gamma_T = Cay(D_{2p}, T)$ are non-isomorphic and cospectral.*

Lemma 3.2. *Let $p \geq 23$ be a prime number and D_{2p} be the dihedral group. If $S' = \{\sigma, \tau\sigma, \tau^5\sigma, \tau^7\sigma, \tau^8\sigma, \tau^{10}\sigma, \tau^{12}\sigma\}$ and $T' = \{\sigma, \tau\sigma, \tau^2\sigma, \tau^5\sigma, \tau^7\sigma, \tau^9\sigma, \tau^{12}\sigma\}$, then the two Cayley graphs $\Gamma_{S'} = Cay(D_{2p}, S')$ and $\Gamma_{T'} = Cay(D_{2p}, T')$ are non-isomorphic and cospectral.*

Proof. We can see that two sets S'_τ and T'_τ are empty. Also, we have $S'_\sigma = \{0, 1, 5, 7, 8, 10, 12\}$ and $T'_\sigma = \{0, 1, 2, 5, 7, 9, 12\}$. By Theorem 2.2, the two Cayley graphs $\Gamma_{S'}$ and $\Gamma_{T'}$ are cospectral, since two multisets $S'_\sigma - S'_\sigma := \{x - y \mid x, y \in S'_\sigma\}$ and $T'_\sigma - T'_\sigma := \{x - y \mid x, y \in T'_\sigma\}$ are equal. Now we prove that the two graphs $\Gamma_{S'}$ and $\Gamma_{T'}$ are not isomorphic. To the contrary, suppose that $\Gamma_{S'}$ and $\Gamma_{T'}$ are isomorphic. Since the dihedral group D_{2p} is CI-group (see Theorem 2.4), there exists an automorphism $\alpha_{s,t} = \alpha \in Aut(D_{2p})$, such that $S' = (T')^\alpha$. Therefore, for some suitable integers t and s, $0 \leq t \leq p-1$ and $1 \leq s \leq p-1$, we have

$$(T'_\sigma)^\alpha = \{t = a_1, t+s = a_2, t+2s = a_3, t+5s = a_4, t+7s = a_5, t+9s = a_6, t+12s = a_7\}, \quad (2)$$

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