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Note

Note on matchings in 3-partite 3-uniform hypergraphs

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ABSTRACT

For a hypergraph H, let $\delta_1(H)$ denote the minimum vertex degree of H, and $\nu(H)$ denote the maximum size of a matching in H. For integers $n \geq m \geq 1$, let

$$d_3(n,m) = \begin{cases} n^2 - (n - \lfloor m/3 \rfloor)(n - \lfloor (m+1)/3 \rfloor) & \text{if } m \neq 1 \pmod 3, \\ n^2 - (n - (m-1)/3)^2 + 1 & \text{if } m = 1 \pmod 3. \end{cases}$$

Let H be a 3-partite 3-uniform hypergraph with n vertices in each partition class. Lo and Markström proved that there exists a positive integer N such that if $n \geq N$ and $\delta_1(H) > d_3(n, n-1)$, then $\nu(H) > n-1$. They also showed that if $n \geq 3^7 m$ and $\delta_1(H) > d_3(n,m)$, then $\nu(H) > m$, and asked whether the condition $n \geq 3^7 m$ can be replaced by $n > m$. In this note, we show that there exists a positive integer n_0 such that if $n \geq n_0$ and $\delta_1(H) > d_3(n,m)$, then $\nu(H) > m$.

1. Introduction

Let k be a positive integer. For a set S, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A *hypergraph* H consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. A hypergraph H is *k-uniform* if $E(H) \subseteq \binom{V(H)}{k}$, and a k-uniform hypergraph is also called a *k-graph*. A k-graph H is *k-partite* if there exists a partition of $V(H)$ into sets $V_1, \ldots, V_k$ (called *partition classes*) such that for any $f \in E(H)$, $|f \cap V_i| = 1$ for $i \in [k] := \{1, \ldots, k\}$. Given $W \subset V(H)$, let $H[W] := \{e \in E(H) \mid e \subseteq W\}$ and $H - W := H[V(H) - W]$.

Let H be a k-graph and $T \subseteq V(H)$. Let $N_H(T) = \{S : S \subseteq V(H) \text{ and } S \cup T \in E(H)\}$. The *degree* of T in H, denoted by $d_H(T)$, is the number of edges of H containing T, i.e., $d_H(T) = |N_H(T)|$. Let l be a nonnegative integer; then $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}\}$ denotes the *minimum l-degree* of H. Hence, $\delta_0(H)$ is the number of edges in H. Note that $\delta_1(H)$ is often called the *minimum vertex* degree of H, and $\delta_{k-1}(H)$ is also known as the *minimum codegree* of H. A *matching* in H is a set of pairwise disjoint edges of H, and it is *perfect* if the union of all edges in the matching is $V(H)$. We use $\nu(H)$ to denote the largest size of a matching in H. A *maximum* matching in H is a matching in H of size $\nu(H)$.

Bollobás, Daykin and Erdős [2] considered minimum vertex degree conditions for the existence of a matching of size $m \geq 1$. They proved that for integers $k \geq 2$ and $m \geq 1$, if H is a k-graph with $|V(H)| = n \geq 2k^2(m+2)$ and $\delta_1(H) > \binom{n-1}{k-1} - \binom{n-m}{k-1}$, then $\nu(H) \geq m$. For 3-graphs, Kühn, Osthus and Treglown [9] proved a stronger result: There exists a positive integer n_0 such that if H is a 3-graph with $|V(H)| = n \geq n_0$, m is an integer with $1 \leq m \leq n/3$, and $\delta_1(H) > \binom{n-1}{2} - \binom{n-m}{2}$, then $\nu(H) \geq m$. For perfect matching, this result was also proved by Khan [5]. Moreover, Khan [6] proved that there exists a positive integer n_1 such that if H is a 4-graph with $|V(H)| = n \geq n_1$, $n \equiv 0 \pmod 4$ and $\delta_1(H) > \binom{n-1}{3} - \binom{\frac{3n}{4}+1}{4}$,

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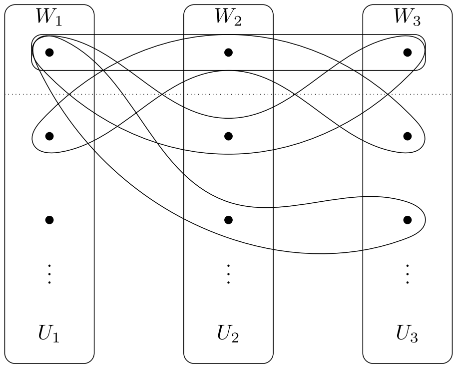

Fig. 1. $H_3(n, m)$.

then H has a perfect matching. Markström and Ruciński [11] gave a lower bound on the minimum l-degree which guarantees the existence of a perfect matching in a k-uniform hypergraph, where $1 \le l < k/2$. This was improved in [8].

There has also been recent interest in matchings in k-partite k-graphs. For a k-partite k-graph H, a set $T \subseteq V(H)$ is said to be *legal* if $|T \cap V_i| \le 1$ for all $i \in [k]$, and *balanced* if $|T \cap V_i| = |T \cap V_j|$ for all $i, j \in [k]$. Thus, if T is not legal in H, then $d_H(T) = 0$. So for integers $0 \le l \le k-1$, let $\delta_l(H) := \min\{d_H(T) : T \in \binom{V(H)}{l}$ and T is legal$\}$. Daykin and Hä ggkvist [3] proved that $\delta_1(H) > \frac{k-1}{k}(n^{k-1} - 1)$ guarantees a perfect matching. This was extended in [4]: if $\delta_d(H) > \frac{k-d}{k}n^{k-d} + kn^{k-d-1}$, then H contains a matching covering all but $k(d-1)$ vertices, and so, a perfect matching for $d = 1$. For constant α and sufficiently large integer n, Pikhurko [12] proved that if for every legal $S \subseteq V_1 \cup \cdots \cup V_l$ and every legal $S' \subseteq V_{l+1} \cup \cdots \cup V_k$,

$$\frac{d_H(S)}{n^{k-l}} + \frac{d_H(S')}{n^l} > 1 + \alpha,$$

then H contains a perfect matching. Aharoni, Georgakopoulos and Sprüssel [1] proved a result about codegrees for perfect matchings in k-partite k-graphs, answering a question of Kühn and Osthus [7].

Theorem 1 (*Aharoni, Georgakopoulos and Sprüssel, [1]*)**.** *Let H be an r-partite r-graph with partition classes $V_1, \ldots, V_r$, each of size n. If for every legal r-tuple f contained in $V - V_1$ we have $deg_H(f) > n/2$ and for every legal $(r-1)$-tuple g contained in $V - V_r$ we have $deg_H(g) \ge n/2$, then H has a perfect matching.*

Lo and Markström [10] showed that if m is a positive integer and H is a k-partite k-graph with n vertices in each partition class such that $n \ge k^7 m$ and $\delta_1(H) > (m - \lceil m/k \rceil - o(1))n^{k-2}$, then $\nu(H) > m$. In fact, an exact bound is given in [10]. In the case when $k = 3$ and $m = n - 1$, Lo and Markström [10] determined the lower bound on $\delta_1(H)$. For integers $n \ge m \ge 1$, let

$$d_3(n, m) = \begin{cases} n^2 - (n - \lfloor m/3 \rfloor)(n - \lfloor (m+1)/3 \rfloor) & \text{if } m \not\equiv 1 \pmod 3, \\ n^2 - (n - (m-1)/3)^2 + 1 & \text{if } m \equiv 1 \pmod 3. \end{cases}$$

Theorem 2 (*Lo and Markström, [10]*)**.** *There exists a positive integer n_1 such that if H is a 3-partite 3-graph with $n \ge n_1$ vertices in each partition class and $\delta_1(H) > d_3(n, n-1)$, then $\nu(H) > n - 1$.*

Lo and Markström [10] constructed 3-partite 3-graphs H with $\nu(H) = m$ and $\delta_1(H) = d_3(n, m)$.

Definition 3. Let $H_3(n; m)$ be the 3-partite 3-graph with partition classes V_1, V_2, V_3 such that each V_i has a partition U_i, W_i with $|W_i| = \lfloor (m+i-1)/3 \rfloor$ for $i \in [3]$, and $E(H_3(n; m))$ consists of all legal 3-subsets of $V(H_3(n; m))$ intersecting $W_1 \cup W_2 \cup W_3$ (see Fig. 1). For later use, let $H'_3(n; m)$ be the 3-partite 3-graph obtained from $H_3(n; m)$ by removing all edges contained in $W_1 \cup W_2 \cup W_3$.

Clearly, $\nu(H_3(n, m)) = m$ (when $n \ge m$). If $m \not\equiv 1 \pmod 3$, $\delta_1(H_3(n; m)) = d_3(n, m)$. If $m = 1 \pmod 3$, $\delta_1(H_3(n; m-1) \cup H') = d_3(n, m) > \delta_1(H_3(n; m))$, where $V(H') = V(H_3(n; m-1))$ and $E(H')$ consists of all legal 3-sets T with $|T \cap \{u_1, u_2, u_3\}| \ge 2$ for some fixed $u_i \in U_i$, $i \in [3]$. Hence the bound in Theorem 2 is tight. Lo and Markström asked the following question (see the paragraph after Corollary 1.7 in [10]).

Question 4 (*Lo and Markström, [10]*)**.** *For integers $n > m \ge 1$, is it true that $\nu(H) > m$ for every 3-partite 3-graph H with each partition class of size n and $\delta_1(H) > d_3(n, m)$?*

Here, we answer Question 4 in the affirmative for large n.

Theorem 5. *There exists a positive integer n_2 such that for integers $n > m \ge 1$, if H is a 3-partite 3-graph with $n \ge n_2$ vertices in each partition class and $\delta_1(H) > d_3(n, m)$, then $\nu(H) > m$.*

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