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The strong chromatic index of $(3, \Delta)$-bipartite graphs

Mingfang Huang [a], Gexin Yu [b], Xiangqian Zhou [c]

[a] Department of Mathematics, School of Science, Wuhan University of Technology, China
[b] Department of Mathematics, The College of William and Mary, Williamsburg, VA 23185, United States
[c] Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, United States

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ABSTRACT

A strong edge-coloring of a graph $G = (V, E)$ is a partition of its edge set E into induced matchings. We study bipartite graphs with one part having maximum degree at most 3 and the other part having maximum degree Δ. We show that every such graph has a strong edge-coloring using at most 3Δ colors. Our result confirms a conjecture of Brualdi and Quinn Massey (1993) for this class of bipartite graphs.

1. Introduction

Graphs in this article are assumed to be simple and undirected. Let G be a simple undirected graph. A *proper edge-coloring* of G is an assignment of colors to the edges such that no two adjacent edges have the same color. Clearly, every coloring class is a matching of G. However, these matchings may not be induced. If one requires each color class to be an induced matching, that leads to the notion of strong edge-coloring, first introduced by Fouquet and Jolivet [5]. A *strong edge-coloring* of a graph G is a proper edge-coloring such that every two edges joined by another edge are colored differently. In a strong edge-coloring, every color class induces a matching. The minimum number of colors required in a strong edge-coloring of G is called the *strong chromatic index* and is denoted by $\chi'_s(G)$.

Let e and e' be two edges of G. We say that e *sees* e' if e and e' are adjacent or share a common adjacent edge. So, equivalently, a strong edge-coloring is an assignment of colors to all edges such that every two edges that can see each other receive distinct colors.

Let Δ be the maximum degree of G and for $u \in V(G)$, let $d_G(u)$ denote the degree of u in the graph G. For each $S \subseteq V(G)$, let $\Delta(S) = \max\{d_G(s) : s \in S\}$. Using greedy coloring arguments, one may easily show that $\chi'_s(G) \le 2\Delta^2 - 2\Delta + 1$ holds for every graph G. Erdős and Nešetřil [3] conjectured the following tighter upper bounds and they also gave examples of graphs that achieve these bounds.

Conjecture 1.1 (*Erdős and Nešetřil [3]*). *For every graph G, the following inequalities hold.*

$$\chi'_s(G) \le \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \text{if } \Delta \text{ is odd.} \end{cases}$$

E-mail address: xiangqian.zhou@wright.edu (X. Zhou).

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In this paper, we study strong edge-coloring of bipartite graphs. Faudree Gyárfás, Schelp, and Tuza [4] conjectured the following.

Conjecture 1.2 (*Faudree et al. [4]*)**.** *For every bipartite graph G, the strong chromatic index of G is at most* Δ^2.

Steger and Yu [7] confirmed Conjecture 1.2 when the maximum degree is at most 3. Let d_A and d_B be two positive integers. A (d_A, d_B)-*bipartite graph* is a bipartite graph with bipartition A and B such that $\Delta(A) \le d_A$ and $\Delta(B) \le d_B$. Brualdi and Quinn Massey [2] strengthened Conjecture 1.2 to the following.

Conjecture 1.3 (*Brualdi and Quinn Massey [2]*)**.** *If G is a* (d_A, d_B)*-bipartite graph, then* $\chi'_s(G) \le d_A d_B$.

Note that, the bounds given in Conjectures 1.2 and 1.3, if proven, would be tight; as the complete bipartite graph $K_{m,n}$ has strong chromatic index mn.

Nakprasit [6] confirmed Conjecture 1.3 for the class of $(2, \Delta)$-bipartite graphs. Recently, Bensmail, Lagoutte, and Valicov [1] proved the following result.

Theorem 1.4 (*Bensmail et al. [1]*)**.** *If G is a* $(3, \Delta)$*-bipartite graph, then* $\chi'_s(G) \le 4\Delta$.

Note that Theorem 1.4 gives a weaker bound than what is given in Conjecture 1.3. In the last section of their paper, the authors of [1] pointed out several possible strategies to improve the bound down to 3Δ. Following their suggestions, we prove the following result.

Theorem 1.5. *If G is a* $(3, \Delta)$*-bipartite graph, then* $\chi'_s(G) \le 3\Delta$.

Our proof scheme is very similar to a scheme used in [1,2,7], first introduced in [7]. The scheme consists of using a matrix to describe a special decomposition of the graph. One minor difference in our approach is that we do not use a matrix, but instead work directly with the decomposition. The main difference in our approach lies in two aspects: the way we choose the decomposition of G and the order in which the edges are colored. Details on each will be presented in Sections 2 and 3, respectively.

The paper is organized as follows. In Section 2, we define a decomposition of G where G is a $(3, \Delta)$-bipartite graph and we also prove some basic properties of the decomposition. The main proof is presented in Section 3. Finally in Section 4 we talk about some possible extensions of our result.

2. A decomposition of *G*

Suppose that G is a $(3, \Delta)$-bipartite graph with bipartition (A, B) with $\Delta(A) \le 3$. Our goal is to show that G has a strong edge-coloring using at most 3Δ colors. Nakprasit's theorem [6] implies that the result holds if $\Delta(A) \le 2$ or $\Delta(B) \le 2$. So we may assume that $\Delta(A) = 3$ and that $\Delta \ge 3$. We may further assume that all vertices of A are of degree exactly 3 (for otherwise, we may add a number of degree-1 vertices to B and increase the degree of every vertex of A to 3).

Now we decompose the graph G into Δ edge-disjoint spanning subgraphs $G_1, G_2, \ldots, G_\Delta$ such that $E = \cup_{i=1}^{\Delta} E(G_i)$, and $d_{G_i}(b) \le 1$ for each $b \in B$ and for each $i \in \{1, 2, \ldots, \Delta\}$. We call such a decomposition a *B-singular decomposition* of G.

Let $G_1, G_2, \ldots, G_\Delta$ be a B-singular decomposition of G. For every vertex $a \in A$ and for every $i \in \{1, 2, \ldots, \Delta\}$, we have that $0 \le d_{G_i}(a) \le 3$. Here we will use the notions of type-1, type-2, and type-3 vertices introduced in [1] and we also require some new notions on the edges of G.

Definition 2.1. Let a be a vertex of A.

- If there exists $1 \le i \le \Delta$ with $d_{G_i}(a) = 3$, then a is called a *type-1 vertex*, and the edges incident to a are called *triplex-edges*.
- If there exists $1 \le i \le \Delta$ with $d_{G_i}(a) = 2$, then a is called a *type-2 vertex*, and the two edges of G_i incident to a are called *paired-edges*, the edge incident to a that is not in G_i is called a *lonely-edge*.
- If there exist distinct i, j, and k such that $d_{G_i}(a) = d_{G_j}(a) = d_{G_k}(a) = 1$, then a is called a *type-3 vertex*, and the edges incident to a are called *dispersed-edges*.

For each $1 \le i \le \Delta$, let H_i be the induced subgraph of G spanned by the endpoints of all lonely-edges of G_i. Note that H_i may contain edges that are not in G_i. Since G is bipartite, a cycle C of H_i must be of even length. Suppose that $|C| = 2k$. Then k may be even or odd. Let $\mathscr{C} = \{C \in C_{2k}: k \text{ is odd and } C \text{ is a cycle in } H_i \text{ for some } 1 \le i \le \Delta\}$. We now choose a special B-singular decomposition $\mathscr{F} = \{G_1, G_2, \ldots, G_\Delta\}$ of G as follows.

(1) First we maximize the number of type-1 vertices;
(2) Subject to (1), we maximize the number of type-2 vertices;
(3) Subject to (1) and (2), we minimize the number of cycles in $\mathscr{C}$.

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