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Note

Counting self-avoiding walks on free products of graphs

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ABSTRACT

The *connective constant* $\mu(G)$ of a graph G is the asymptotic growth rate of the number σ_n of self-avoiding walks of length n in G from a given vertex. We prove a formula for the connective constant for free products of quasi-transitive graphs and show that $\sigma_n \sim A_G\mu(G)^n$ for some constant A_G that depends on G. In the case of products of finite graphs $\mu(G)$ can be calculated explicitly and is shown to be an algebraic number.

1. Introduction

An *n-step self-avoiding walk* (SAW) on a rooted graph G is a path of length n starting in the root o where no vertex appears more than once. The *connective constant* describes the asymptotic growth of the number σ_n of self-avoiding walks of length n on a given graph G; it is defined as

$$\mu(G) = \lim_{n\to\infty} \sigma_n^{1/n}, \tag{1}$$

provided the limit exists. In fact, if the graph is (quasi-)transitive the existence of the limit is guaranteed by subadditivity of σ_n (Hammersley [14]). In the case of a finite graph $\mu(G)$ trivially equals zero. In the following we will just write $\mu = \mu(G)$.

Self-avoiding walks were originally introduced on Euclidean lattices as a model for long polymer chains. But in recent years, the study on hyperbolic lattices (Madras and Wu [19], Swierczak and Guttmann [22]), on one-dimensional lattices (Alm and Janson [1]), and on general graphs and Cayley graphs (Grimmett and Li [8–12]) has received increasing attention from physicists and mathematicians alike. In particular, the connective constant of a d-regular vertex-transitive simple graph is shown to be bounded below by $\sqrt{d-1}$ [11]. However, exact values of the connective constant are only known for a small class of non-trivial graphs, namely ladder graphs [1], the hexagonal lattice where the connective constant equals $\sqrt{2+\sqrt{2}}$ (Duminil-Copin and Smirnov [5]), and the (3.12^2) lattice (Jensen and Guttmann [16]).

In the present paper we use generating functions, in particular the function $\mathcal{M}(z) := \sum_{n\geq 0}\sigma_n \cdot z^n$, to derive formulas for the connective constants for free products of graphs. Generating functions have already played an important role in the theory of self-avoiding walks, e.g. see [1] and Bauerschmidt, Duminil-Copin, Goodman and Slade [2]. The approach of this note is to show that the involved generating functions satisfy a functional equation of the same type as generating functions for ordinary random walks. As a consequence the connective constant can be expressed as the smallest positive root of an equation, see Theorem 3.3. In the case of products of finite graphs this equation can be solved explicitly and it is shown that μ is an algebraic number, see Corollary 3.4.

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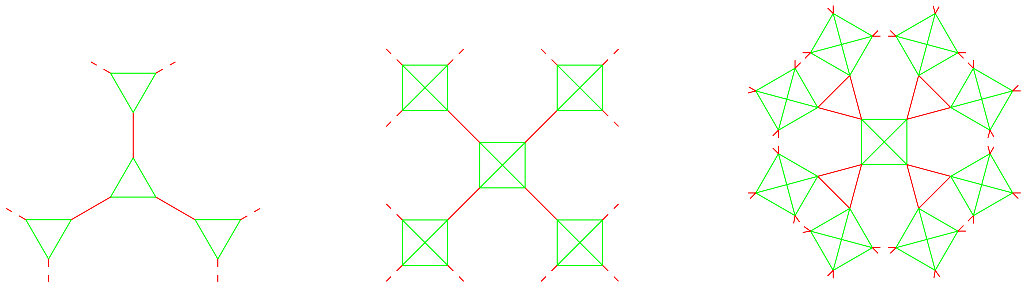

(a) $K_2 * K_3 : \mu \approx 1.7692.$ [1] (b) $K_2 * K_4 : \mu \approx 2.5163.$ (c) $K_3 * K_4 : \mu \approx 3.6151.$

Fig. 1. Examples for free products of complete graphs K_n.

It is widely believed that on $\mathbb{Z}^d$, $d \neq 4$, there is a critical exponent γ depending on d such that $\sigma_n \sim A\mu^n n^{1-\gamma}$, where A is a constant depending on the underlying graph. However, this behaviour is rigorously proven only on lattices in dimensions $d \geq 5$ (Hara and Slade [15]), on ladder graphs [1], and (in a weaker form) on regular tessellations of the hyperbolic lattice [19]. In this note we prove that $\sigma_n \sim A\mu(G)^n n^{1-\gamma}$ with $\gamma = 1$ for free products of graphs, see Theorem 4.1. This result supports the conjecture proposed in [19] that SAW on nonamenable graphs exhibits a mean-field behaviour, i.e., that $\gamma = 1$.

The study of stochastic processes on free products has a long and fruitful history. The probably earliest publications on free products of rooted graphs are due to Teh and Gan [23] and Znoĭko [27]. For further details on products of graphs, we refer to Hammack, Imrich and Klavžar [13]; for the special case of free products of groups, we refer to Lyndon and Schupp [18]. In most of the works generating function techniques played an important role, e.g. Woess [26]. Similar techniques to the ones we use for rewriting generating functions in terms of functions on the factors of the free product were introduced independently and simultaneously in Cartwright and Soardi [4], McLaughlin [20], Voiculescu [24] and Woess [25]. Free products owe some of their importance to Stalling's Splitting Theorem which states that a finitely generated group has more than one (geometric) end if and only if it admits a nontrivial decomposition as an amalgamated free product or an HNN-extension over a finite subgroup. Furthermore, in many cases exact calculations are possible on free products whereas so far these kind of results seem out of reach on one-ended nonamenable graphs. For example, the spectral radius of random walks (we refer e.g. to [26]) and the critical percolation probability p_c (Špakulová [17,21]) can be calculated on free products but still constitute a big challenge on one-ended nonamenable graphs.

2. Free products of graphs

Let $G = (V, E, o)$ be a rooted graph with vertex set V, edge set E, and distinguished vertex o. The graphs considered in this note are locally finite, connected and simple, i.e. no multiple edges between any pair of vertices. An (undirected) edge $e \in E$ with endpoints $u, v \in V$ is noted as $e = \langle u, v\rangle$. Two vertices u, v are called *adjacent* if there exists an edge $e \in E$ such that $e = \langle u, v\rangle$; in this case we write $u \sim v$. The distinguished vertex $o \in V$ is called the root (or origin) of the graph. A *path* of length $n \in \mathbb{N}$ on a graph G is a sequence of vertices $[v_0, v_1, \ldots, v_n]$ such that $v_{i-1} \sim v_i$ for $i \in \{1, \ldots, n\}$. We recall that a graph is called *quasi-transitive* if its automorphism group acts quasi-transitive, i.e. with finitely many orbits. In particular, every finite graph is quasi-transitive.

We now give a standard definition of free products of graphs (see e.g. [26]). Let $r \in \mathbb{N}$ with $r \geq 2$ and set $\mathcal{I} := \{1, \ldots, r\}$. Let $G_1 = (V_1, E_1, o_1), \ldots, G_r = (V_r, E_r, o_r)$ be a finite family of undirected, connected, quasi-transitive, rooted graphs with vertex sets V_i, edge sets E_i and roots o_i for $1 \leq i \leq r$. We assume also that $|V_i| \geq 2$ for every $i \in \mathcal{I}$ and that the vertex sets are distinct.

Let $V_i^\times := V_i \setminus \{o_i\}$ for every $i \in \mathcal{I}$ and define the function $\tau : \bigcup_{i\in\mathcal{I}} V_i^\times \to \mathcal{I}$ by $\tau(x) := i$ if $x \in V_i^\times$. Define

$$V := V_1 * \cdots * V_r = \left\{ x_1 x_2 \ldots x_n \mid n \in \mathbb{N}, x_i \in \bigcup_{j\in\mathcal{I}} V_j^\times, \tau(x_i) \neq \tau(x_{i+1}) \right\} \cup \{o\},$$

which is the set of 'words' over the alphabet $\bigcup_{i\in\mathcal{I}} V_i^\times$ such that no two consecutive letters come from the same $V_i^\times$. The empty word in V is denoted by o. We extend the function τ on V by setting $\tau(x_1 \ldots x_n) := \tau(x_n)$ for $x_1 x_2 \ldots x_n \in V$. On the set V we have a partial word composition law: if $x = x_1 \ldots x_m, y = y_1 \ldots y_n \in V$ with $\tau(x_m) \neq \tau(y_1)$ then xy stands for the concatenation of x and y, which is again an element of V. In particular, for $x = x_1 \ldots x_n \in V$, if $x_1 \notin V_i^\times$ then we set $o_i x := x$, and if $x_n \notin V_i^\times$ then we set $x o_i := x$. Additionally, we set $xo := ox := x$. We regard each V_i as a subset of V, identifying each o_i with o.

[1] The exact value of μ is $6\left(-2 + \sqrt[3]{46 - 6\sqrt{57}} + \sqrt[3]{46 + 6\sqrt{57}}\right)^{-1}$.