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Maximal m-distance sets containing the representation of the Hamming graph $H(n, m)$

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ABSTRACT

A set X in the Euclidean space $\mathbb{R}^d$ is an *m-distance set* if the set of Euclidean distances between two distinct points in X has size m. An m-distance set X in $\mathbb{R}^d$ is *maximal* if there does not exist a vector $\mathbf{x}$ in $\mathbb{R}^d$ such that the union of X and $\{\mathbf{x}\}$ still has only m distances. Bannai et al. (2012) investigated maximal m-distance sets that contain the Euclidean representation of the Johnson graph $J(n, m)$. In this paper, we consider the same problem for the Hamming graph $H(n, m)$. The Euclidean representation of $H(n, m)$ is an m-distance set in $\mathbb{R}^{m(n-1)}$. We prove that if the representation of $H(n, m)$ is not maximal as an m-distance set for some m, then the maximum value of n is $m^2 + m - 1$. Moreover we classify the largest m-distance sets that contain the representation of $H(n, m)$ for $n \geq 2$ and $m \leq 4$. We also classify the maximal 2-distance sets that are in $\mathbb{R}^{2n-1}$ and contain the representation of $H(n, 2)$ for $n \geq 2$.

1. Introduction

A subset X of the Euclidean space $\mathbb{R}^d$ is an *m-distance set* if the size of the set of distances between two distinct points in X is equal to m. The size of an m-distance set is bounded above by $\binom{d+m}{m}$ [3]. One of major problems is to find the maximum possible cardinality of an m-distance set for given m and d. The largest 1-distance set in $\mathbb{R}^d$ is the regular simplex for $d \geq 1$, and it has $d + 1$ points. Largest 2-distance sets in $\mathbb{R}^d$ are classified for $d \leq 7$ [6,11]. Lisoněk [11] constructed a largest 2-distance set in $\mathbb{R}^8$, which is the only known set attaining the bound $|X| \leq \binom{d+m}{m}$ for $m \geq 2$. Largest m-distance sets in $\mathbb{R}^2$ are classified for $m \leq 5$ [7,12,13]. Two largest 6-distance sets are known [15]. Tables 1, 2 show the cardinalities $|X|$ of largest distance sets X, and the number # of the sets, up to isometry. The largest 3-distance set in $\mathbb{R}^3$ is the vertex set of the icosahedron [14].

The Euclidean representation $\tilde{J}(n, m)$ of the Johnson scheme $J(n, m)$ is the subset of $\mathbb{R}^n$ consisting of all vectors with 1's in m coordinates and 0's elsewhere. The set $\tilde{J}(n, m)$ with $n \geq 2m$ can be interpreted as an m-distance set in $\mathbb{R}^{n-1}$ because the sum of entries of each element is m. The largest known m-distance sets in $\mathbb{R}^{n-1}$ are mostly $\tilde{J}(n, m)$. An m-distance set X in $\mathbb{R}^n$ is *maximal* if there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $X \cup \{\mathbf{x}\}$ is still m-distance. Bannai, Sato, and Shigezumi [4] investigated maximal m-distance sets that are in $\mathbb{R}^{n-1}$ and contain $\tilde{J}(n, m)$. They gave a necessary and sufficient condition for $\tilde{J}(n, m)$ to be a maximal m-distance set in $\mathbb{R}^{n-1}$, and classified the largest m-distance sets containing $\tilde{J}(n, m)$ for $n \geq 2$ and $m \leq 5$, except

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Table 1
$m = 2$.

d	2	3	4	5	6	7	8
$\|X\|$	5	6	10	16	27	29	45
#	1	6	1	1	1	1	≥1

Table 2
$d = 2$.

m	2	3	4	5	6
$\|X\|$	5	7	9	12	13
#	1	2	4	1	≥2

Table 3
$m = 2$.

n	5
d	8
$\|X\|$	40

Table 4
$m = 3$.

n	3	5	9	11
d	6	12	24	30
$\|X\|$	40	200	981	1451

Table 5
$m = 4$.

n	2	3	5	6	7	9	11	13	14	19
d	4	8	16	20	24	32	40	48	52	72
$\|X\|$	25	222	1600	2004	3390	8829	16566	29056	39417	133381

for $(n, m) = (9, 4)$. The case $(n, m) = (9, 4)$ is solved in [1]. This construction of distance sets might be possible for other association schemes. In this paper we consider the Hamming scheme $H(n, m)$.

Let $F_n = \{1, \dots, n\}$, $\mathbf{x} = (x_1, \dots, x_m) \in F_n^m$, and $\mathbf{y} = (y_1, \dots, y_m) \in F_n^m$. The Hamming distance of $\mathbf{x}$ and $\mathbf{y}$ is defined to be $h(\mathbf{x}, \mathbf{y}) = |\{i : x_i \neq y_i\}|$. The Hamming scheme $H(n, m)$ is an association scheme $(F_n^m, \{R_0, \dots, R_m\})$, where $R_i = \{(\mathbf{x}, \mathbf{y}) : h(\mathbf{x}, \mathbf{y}) = i\}$. Let $\varphi : F_n^m \to \mathbb{R}^{mn}$ be the embedding defined by

$$\varphi : \mathbf{x} = (x_1, \dots, x_m) \mapsto \tilde{\mathbf{x}} = \sum_{i=1}^{m} \mathbf{e}_{(i-1)n+x_i},$$

where $\{\mathbf{e}_1, \dots, \mathbf{e}_{mn}\}$ is the standard basis of $\mathbb{R}^{mn}$. Let $\tilde{H}(n, m)$ denote the image of φ. Note that $h(\mathbf{x}, \mathbf{y}) = k$ for $\mathbf{x}, \mathbf{y} \in H(n, m)$ if and only if $d(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \sqrt{2k}$ for $\tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \tilde{H}(n, m)$, where $d(\,,\,)$ is the Euclidean distance. Let $\mathbf{j}_k$ denote the vector

$$\mathbf{j}_k = \sum_{i=(k-1)n+1}^{kn} \mathbf{e}_i.$$

Every vector in $\tilde{H}(n, m)$ is perpendicular to $\mathbf{j}_k$ for $k \in \{1, \dots, m\}$. We can therefore interpret $\tilde{H}(n, m)$ as an m-distance set in $\mathbb{R}^{m(n-1)}$. We consider maximal m-distance sets that are in $\mathbb{R}^{m(n-1)}$ and contain $\tilde{H}(n, m)$.

This paper is summarized as follows. In Section 2, we give some notation, and determine the coordinates of a vector $\mathbf{x}$ when $\mathbf{x}$ can be added to $\tilde{H}(n, m)$ while maintaining m-distance. In Section 3, the maximal 2-distance sets containing $\tilde{H}(n, 2)$ are classified by an explicit way. In Section 4, we give a necessary and sufficient condition for $\tilde{H}(n, m)$ to be maximal as an m-distance set. Moreover, we prove that if $\tilde{H}(n, m)$ is not maximal as an m-distance set for some m, then the maximum value of n is equal to $m^2 + m - 1$. In Section 5, we classify the largest m-distance sets that are in $\mathbb{R}^{m(n-1)}$ and contain $\tilde{H}(n, m)$ for $n \geq 2$ and $m \leq 4$. Tables 3–5 show the maximum cardinalities $|X|$ and dimension $d = m(n-1)$. In Section 6, we classify maximal 2-distance sets that are in $\mathbb{R}^{2(n-1)+1}$ and contain $\tilde{H}(n, 2)$.

2. Vectors that can be added to $\tilde{H}(n, m)$

First we give some notation. For real numbers $x_1, \dots, x_n$ and natural numbers $\lambda_1, \dots, \lambda_n$, we use the notation

$$(x_1^{\lambda_1}, \dots, x_n^{\lambda_n}) = (\underbrace{x_1, \dots, x_1}_{\lambda_1}, \dots, \underbrace{x_n, \dots, x_n}_{\lambda_n}) \in \mathbb{R}^{\lambda_1 + \cdots + \lambda_n},$$