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On acyclic edge-coloring of complete bipartite graphs

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ABSTRACT

An acyclic edge-coloring of a graph is a proper edge-coloring without bichromatic (2-colored) cycles. The acyclic chromatic index of a graph G, denoted by $a'(G)$, is the least integer k such that G admits an acyclic edge-coloring using k colors. Let $\Delta = \Delta(G)$ denote the maximum degree of a vertex in a graph G. A complete bipartite graph with n vertices on each side is denoted by $K_{n,n}$. Basavaraju, Chandran and Kummini proved that $a'(K_{n,n}) \geq n + 2 = \Delta + 2$ when n is odd. Basavaraju and Chandran provided an acyclic edge-coloring of $K_{p,p}$ using $p + 2$ colors and thus establishing $a'(K_{p,p}) = p + 2 = \Delta + 2$ when p is an odd prime. The main tool in their approach is perfect 1-factorization of $K_{p,p}$. Recently, following their approach, Venkateswarlu and Sarkar have shown that $K_{2p-1,2p-1}$ admits an acyclic edge-coloring using $2p + 1$ colors which implies that $a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$, where p is an odd prime. In this paper, we generalize this approach and present a general framework to possibly get an acyclic edge-coloring of $K_{n,n}$ which possesses a perfect 1-factorization using $n + 2 = \Delta + 2$ colors. In this general framework, using number theoretic techniques, we show that K_{p^2,p^2} admits an acyclic edge-coloring with $p^2 + 2$ colors and thus establishing $a'(K_{p^2,p^2}) = p^2 + 2 = \Delta + 2$ when p is an odd prime.

1. Introduction

Let $G = (V, E)$ be a finite and simple graph. A *proper edge-coloring* of G is an assignment of colors to the edges so that no two adjacent edges have the same color. So it is a map $\theta : E \to \mathcal{C}$ with $\theta(e) \neq \theta(f)$ for any adjacent edges $e, f \in E$, where $\mathcal{C}$ is the set of colors. The *chromatic index*, denoted by $\chi'(G)$, is the least integer k such that G admits a proper edge-coloring using k colors. A proper coloring of G is *acyclic* if there is no two-colored cycle in G. The *acyclic edge chromatic number* (also called *acyclic chromatic index*), denoted by $a'(G)$, is the least integer k such that G admits an acyclic edge-coloring using k colors. The notion of acyclic coloring was first introduced by Grünbaum [14] in 1973, and the concept of acyclic edge-coloring was first studied by Fiamčík [12]. Let $\Delta = \Delta(G)$ be the maximum degree of a vertex in G. It is obvious that any proper edge-coloring requires at least Δ colors. Vizing [23] proved that there always exists a proper edge-coloring with $\Delta + 1$ colors. Since any acyclic edge coloring is proper, we must have $a'(G) \geq \chi'(G) \geq \Delta$. In 1978, Fiamčík [12] (also independently Alon, Sudakov and Zaks [3]) posed the following conjecture.

Conjecture 1.1. *For any graph G, $a'(G) \leq \Delta + 2$.*

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In [3] it was proved that there exists a constant c such that $a'(G) \leq \Delta+2$ for any graph with girth at least $c\Delta \log \Delta$. It was also proved in [3] that $a'(G) \leq \Delta + 2$ for almost all Δ-regular graphs. Later Něsetřil and Wormald [19] improved this bound and showed that $a'(G) \leq \Delta + 1$ for a random regular graph G. In another direction, there have been many results giving upper bounds on $a'(G)$ for an arbitrary graph G. For example, Alon, McDiarmid and Reed [2] proved that $a'(G) \leq 64\Delta$. Molloy and Reed [16] improved this bound and showed that $a'(G) \leq 16\Delta$. Recently, Ndreca et al. [20] obtained $a'(G) \leq \lceil 9.62\Delta \rceil$ and then Esperet et al. [11] have shown $a'(G) \leq 4\Delta - 4$. This has been recently improved by Giota et al. [13], who showed that $a'(G) \leq \lceil 3.74(\Delta - 1) \rceil + 1$. The recent improvements rely on various algorithmic versions of Lovász Local Lemma. The acyclic edge-coloring of planar graphs has been deeply studied in recent years. See [24, Section 3.3] for a nice account of recent results.

Conjecture 1.1 was shown to be true for some special classes of graphs. Burnstein [9] showed that $a'(G) \leq 5$ when $\Delta = 3$. Hence the conjecture is true when $\Delta \leq 3$. In fact it was shown in [4] that $a'(G) \leq 4$ when $\Delta(G) \leq 3$ and $G \notin \{K_4, K_{3,3}\}$. Muthu, Narayanan and Subramanian proved that the conjecture holds true for grid-like graphs [17] and outerplanar graphs [18]. It has been observed that determining $a'(G)$ is a hard problem from both theoretical and algorithmic points of view [24, p. 2119]. In fact, we do not yet know the values of $a'(G)$ for some simple and highly structured graphs like complete graphs and complete bipartite graphs in general. Fortunately, we can get the exact value of $a'(G)$ for some cases of complete bipartite graphs, thanks to the perfect 1-factorization.

Let $K_{n,n}$ be the complete bipartite graph with n vertices on each side. The complete bipartite graph $K_{n,n}$ is said to have a perfect 1-factorization if the edges of $K_{n,n}$ can be decomposed into n disjoint perfect matchings such that the union of any two perfect matchings gives a Hamiltonian cycle (see Section 2 for more details). It is known that when $n \in \{p, 2p-1, p^2\}$, where p is an odd prime, or $n < 50$ and odd, then $K_{n,n}$ has a perfect 1-factorization (see [8]). One can easily see that if $K_{n,n}$ has a perfect 1-factorization then $a'(K_{n-2,n-2}) \leq a'(K_{n-1,n-1}) \leq n$. And also we have the following result due to Basavaraju, Chandran and Kummini [6].

Theorem 1.1. *$a'(K_{n,n}) \geq n + 2 = \Delta + 2$, when n is odd.*

Hence $a'(K_{n-2,n-2}) = n = \Delta + 2$ when $n \in \{p, 2p-1, p^2\}$. By a result of Guldan [15, Corollary 1], we can also get $a'(K_{n-1,n-1}) = n = \Delta + 1$ when $n \in \{p, 2p-1, p^2\}$.

The main idea here is to give different colors to the edges in different 1-factors in $K_{n,n}$, and removal of (one) two vertices on each side and their associated edges gives the required edge-coloring of $(K_{n-1,n-1})\,K_{n-2,n-2}$ with n colors. But a different approach is needed to deal with $K_{n,n}$ when $n \in \{p, 2p-1, p^2\}$. In 2009, Basavaraju and Chandran [5] proved that $a'(K_{p,p}) = p + 2 = \Delta + 2$ for any odd prime p. We can view their approach as follows: suitably pick one edge from each 1-factor and partition these edges into two groups and each group can possibly be assigned a different color to get the required result. Following this approach, Venkateswarlu and Sarkar have recently shown that $a'(K_{2p-1,2p-1}) = 2p+1 = \Delta+2$ for any odd prime p [22]. In this paper we view this approach in a more general setting and propose a general framework for the proof. The only remaining infinite class of complete bipartite graphs that are known to have a perfect 1-factorization is K_{p^2,p^2}, where p is an odd prime. In this general framework, we provide an acyclic edge-coloring of K_{p^2,p^2} using $p^2 + 2$ colors when p is an odd prime. Therefore we state our main result as follows.

Theorem 1.2. *$a'(K_{p^2,p^2}) = p^2 + 2 = \Delta + 2$, where p is an odd prime.*

Therefore the acyclic chromatic index is equal to $\Delta+2$ for all the three known infinite classes of complete bipartite graphs having a perfect 1-factorization, and Conjecture 1.1 holds true for such graphs.

In the next section we discuss some preliminaries and in Section 3 we present a general framework to possibly get an acyclic edge-coloring of $K_{n,n}$ which possesses a perfect 1-factorization using $n + 2$ colors. Then we present a proof of Theorem 1.2 in this framework in Section 4.

2. Preliminaries

Let $n\,(\geq 2)$ be an integer. We treat elements of the ring $\mathbb{Z}_n$ as integers in the range $\{0, 1, \ldots, n-1\}$. We denote the complete bipartite graph $K_{n,n}$ as $G = (V \cup V', E)$ with $|V| = |V'| = n$ and $E = \{(v \mapsto v') : v \in V \text{ and } v' \in V'\}$. We use $\mapsto$ to define edges though our graph $K_{n,n}$ is undirected. This is only for ease of presentation in associating a perfect matching in $K_{n,n}$ with a permutation of the label set $I(=\{0, 1, \ldots, n-1\})$, which we discuss below. Accordingly, the use of arrows in Figs. 1 and 2 is to explicitly emphasize the correspondence between a perfect matching and its associated permutation map. We use the terms 'composition' and 'product' of permutations interchangeably. Note also that a permutation can be decomposed as a product of disjoint cycles uniquely (up to a reordering of the cycles and cyclic rotation of the elements within a cycle) and it is called a disjoint cycle decomposition. We use $\sqcup$ (instead of the usual union notation $\cup$) to signify union of 'disjoint' sets.

2.1. Perfect matchings and perfect 1-factorizations

A *matching* in a graph is a set of edges without common vertices, and a *perfect matching* is a matching which matches all vertices of the graph. In the case of complete bipartite graph $K_{n,n}$, a perfect matching $M \subset E$ is a set of n edges satisfying: