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Bounds on the exponential domination number

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As a natural variant of domination in graphs, Dankelmann et al. (2009) introduce exponential domination, where vertices are considered to have some dominating power that decreases exponentially with the distance, and the dominated vertices have to accumulate a sufficient amount of this power emanating from the dominating vertices. More precisely, if S is a set of vertices of a graph G, then S is an exponential dominating set of G if $\sum_{v\in S}\left(\frac{1}{2}\right)^{\mathrm{dist}_{(G,S)}(u,v)-1} \geq 1$ for every vertex u in $V(G)\setminus S$, where $\mathrm{dist}_{(G,S)}(u,v)$ is the distance between $u \in V(G)\setminus S$ and $v \in S$ in the graph $G-(S\setminus\{v\})$. The exponential domination number $\gamma_e(G)$ of G is the minimum order of an exponential dominating set of G.

Dankelmann et al. show

$$\frac{1}{4}(d+2) \leq \gamma_e(G) \leq \frac{2}{5}(n+2)$$

for a connected graph G of order n and diameter d. We provide further bounds and in particular strengthen their upper bound. Specifically, for a connected graph G of order n, maximum degree Δ at least 3, and radius r, we show

$$\begin{aligned}\gamma_e(G) &\geq \left(\frac{n}{13(\Delta-1)^2}\right)^{\frac{\log_2(\Delta-1)+1}{\log_2^2(\Delta-1)+\log_2(\Delta-1)+1}},\\ \gamma_e(G) &\leq 2^{2r-2}, \text{ and}\\ \gamma_e(G) &\leq \frac{43}{108}(n+2).\end{aligned}$$

1. Introduction

We consider finite, simple, and undirected graphs, and use standard notation and terminology.

A set D of vertices of a graph G is *dominating* if every vertex not in D has a neighbor in D. The *domination number* $\gamma(G)$ of G, defined as the minimum cardinality of a dominating set, is one of the most well studied quantities in graph theory [16]. As a natural variant of this classical notion, Dankelmann et al. [7] introduce exponential domination, where vertices are considered to have some dominating power that decreases exponentially with the distance, and the dominated vertices have to accumulate a sufficient amount of this power emanating from the dominating vertices. As a motivation of their model they mention information dissemination within social networks, where the impact of information decreases every time it is passed on.

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Before giving the precise definitions for exponential domination, we mention three closely related well studied notions. A set D of vertices of a graph G is *k-dominating* for some positive integer k, if every vertex not in D has at least k neighbors in D [4,5,8,11–13,15,19]. A set D of vertices of a graph G is *distance-k-dominating* for some positive integer k, if for every vertex not in D, there is some vertex in D at distance at most k [1–3,14,18,20]. Finally, in *broadcast domination* [6,9,10,17], each vertex v is assigned an individual dominating power $f(v)$ and dominates all vertices at distance at least 1 and at most $f(v)$. Exponential domination shares features with these three notions; similarly as in k-domination, several vertices contribute to the domination of an individual vertex, similarly as in distance-k-domination, vertices dominate others over some distance, and similarly as in broadcast domination, different dominating vertices contribute differently to the domination of an individual vertex depending on the relevant distances.

We proceed to the precise definitions, and also recall some terminology.

Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order $n(G)$ of G is the number of vertices of G, and the size $m(G)$ of G is the number of edges of G. For two vertices u and v of G, let $\mathrm{dist}_G(u, v)$ be the distance in G between u and v, which is the minimum number of edges of a path in G between u and v. If no such path exists, then let $\mathrm{dist}_G(u, v) = \infty$. An endvertex is a vertex of degree at most 1. For a rooted tree T, and a vertex u of T, let T_u denote the subtree of T rooted in u that contains u as well as all descendants of u. A leaf of a rooted tree is a vertex with no children. For non-negative integers $d_0, d_1, \ldots, d_k$, let $T(d_0, d_1, \ldots, d_k)$ be the rooted tree of depth k in which all vertices at distance i from the root have exactly d_i children for every i with $0 \le i \le k$. A rooted tree is binary if every vertex has at most two children, and a binary tree is full if every vertex other than the leaves has exactly two children. For a positive integer k, let $[k]$ be the set of positive integers less or equals to k.

Let S be a set of vertices of G. For two vertices u and v of G with $u \in S$ or $v \in S$, let $\mathrm{dist}_{(G,S)}(u, v)$ be the minimum number of edges of a path P in G between u and v such that S contains exactly one endvertex of P and no internal vertex of P. If no such path exists, then let $\mathrm{dist}_{(G,S)}(u, v) = \infty$. Note that, if u and v are distinct vertices in S, then $\mathrm{dist}_{(G,S)}(u, u) = 0$ and $\mathrm{dist}_{(G,S)}(u, v) = \infty$.

For a vertex u of G, let

$$w_{(G,S)}(u) = \sum_{v \in S} \left(\frac{1}{2}\right)^{\mathrm{dist}_{(G,S)}(u,v)-1},$$

where $\left(\frac{1}{2}\right)^{\infty} = 0$. Note that $w_{(G,S)}(u) = 2$ for $u \in S$.

If $w_{(G,S)}(u) \ge 1$ for every vertex u of G, then S is an *exponential dominating set* of G. The *exponential domination number* $\gamma_e(G)$ is the minimum order of an exponential dominating set of G, and an exponential dominating set of G of order $\gamma_e(G)$ is *minimum*. By definition, every dominating set is also an exponential dominating set, which implies $\gamma_e(G) \le \gamma(G)$ for every graph G.

The following summarizes the main results of Dankelmann et al. [7].

Theorem 1 (*Dankelmann et al. [7]*)**.** *If G is a connected graph of diameter* $\mathrm{diam}(G)$*, then*

$$\frac{1}{4}(\mathrm{diam}(G) + 2) \le \gamma_e(G) \le \frac{2}{5}(n(G) + 2).$$

Dankelmann et al. [7] discuss the tightness of their bounds. They show that the lower bound is satisfied with equality for the path P_n of order n with $n \equiv 2 \bmod 4$, and they construct a sequence of trees T for which $\frac{\gamma_e(T)}{n(T)+2}$ tends to $\frac{3}{8}$. Finally, they describe one specific tree T with $\frac{\gamma_e(T)}{n(T)+2} = \frac{144}{377} \approx 0.382$, and ask whether there are trees T with $\frac{144}{377} < \frac{\gamma_e(T)}{n(T)+2} \le \frac{2}{5}$.

Note that the lower bound in Theorem 1 implies $\gamma_e(G) = \Omega(\log n(G))$ for graphs G of bounded maximum degree, because the diameter of such graphs is $\Omega(\log n(G))$. Our first result is a polynomial, and not just logarithmic, lower bound.

Theorem 2. *If G is a graph of maximum degree $\Delta(G)$ at least 3, then*

$$\gamma_e(G) \ge \left(\frac{n(G)}{13(\Delta(G) - 1)^2}\right)^{\frac{\log_2(\Delta(G)-1)+1}{\log_2^2(\Delta(G)-1)+\log_2(\Delta(G)-1)+1}}.$$

As our second result, we show that $\gamma_e(G)$ is not only lower bounded but in fact also upper bounded in terms of the diameter of G, or rather the radius of G.

Theorem 3. *If G is a connected graph of radius* $\mathrm{rad}(G)$ *at least* 1*, then* $\gamma_e(G) \le 2^{2\mathrm{rad}(G)-2}$.

Surprisingly, the bound in Theorem 3 is tight as we show by constructing a suitable example.

As our third result, we improve the upper bound in Theorem 1 as follows.

Theorem 4. *If G is a connected graph, then* $\gamma_e(G) \le \frac{43}{108}(n(G) + 2)$.

Note that $\frac{43}{108} \approx 0.398$.

All proofs and further discussion are postponed to the next section.