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# On a construction using commuting regular graphs

Marius Somodi *, Katie Burke, Jesse Todd

*Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614, USA*

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ABSTRACT

We consider a construction using a pair of commuting regular graphs that generalizes the constructions of the middle, total, and quasitotal graphs. We derive formulae for the characteristic polynomials of the adjacency and Laplacian matrices and for the Ihara zeta function of the resulting graph. Using these formulae, we express the number of spanning trees and the Kirchhoff index of the resulting graph in terms of the Laplacian spectra of the two regular graphs used in the construction.



## 1. Introduction

Let $G$ be a & simple graph of order $n$, whose vertices are labeled, in some order, with numbers $1, \ldots, n$. We recall that the adjacency matrix of $G$ is an $n \times n$ matrix $\mathbf{A} = (a_{ij})$, where $a_{ij} = 1$, if vertices $i$ and $j$ are adjacent in $G$, and $a_{ij} = 0$, otherwise. The Laplacian matrix of $G$ is the matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$, where $\mathbf{D}$ denotes the degree matrix of $G$, i.e. the matrix $\mathbf{D} = \mathrm{diag}(d_1, \ldots, d_n)$, where $d_i$ denotes the degree of vertex $i$.

Two graphs of the same order, $G$ and $G'$, are said to commute if one can label the vertices of $G$ and $G'$ such that their adjacency matrices commute, i.e. $\mathbf{AA}' = \mathbf{A}'\mathbf{A}$, where $\mathbf{A}$ and $\mathbf{A}'$ denote the adjacency matrices of $G$ and $G'$, respectively. Properties and examples of commuting graphs can be found in [1,2,8,15]. We recall that connected regular graphs only commute with regular graphs, the complete graph $K_n$ commutes with any regular graph of the same order, and the complete bipartite graph $K_{n,n}$ commutes with any regular subgraph of the same order (see Propositions 2.3.6 and 2.3.7 from [15]).

Starting with $G$ as a base graph, one can construct various transformation graphs e.g. the line graph, the middle graph, the total graph, or the quasitotal graph of $G$ [3,13,14,22]. For the reader's convenience, we recall how these graphs are constructed:

*Line graph $L(G)$*: $V(L(G)) = E(G)$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ have a vertex in common.

*Middle graph $M(G)$*: $V(M(G)) = V(G) \cup E(G)$. $E(M(G)) = E(L(G)) \cup \{ve : e \in E(G), v \in V(G) \text{ are incident in } G\}$.

*Total graph $T(G)$*: $V(T(G)) = V(G) \cup E(G)$. $E(T(G)) = E(M(G)) \cup \{vv' : v, v' \in V(G) \text{ are adjacent in } G\}$.

*Quasitotal graph $QT(G)$*: $V(QT(G)) = V(G) \cup E(G)$. $E(QT(G)) = E(M(G)) \cup \{vv' : v, v' \in V(G) \text{ are not adjacent in } G\}$.

* Corresponding author.
*E-mail address:* marius.somodi@uni.edu (M. Somodi).

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The constructions of the middle, total, and quasitotal graphs of $G$ are particular cases of the following general construction: let $G$ and $G'$ be two simple graphs that have the same order $n$. Label the vertices of $G$ and $G'$, in some order, with numbers $1, \ldots, n$. Construct a new graph, denoted $G \ltimes G'$, as follows: $V(G \ltimes G') = V(G) \cup E(G)$ and $E(G \ltimes G') = E(M(G)) \cup \{ij : i, j \in V(G'), ij \in E(G')\}$.

We will refer to a graph $G \ltimes G'$ as an *overlay* of $G$ and $G'$. Note that, in general, the operation $\ltimes$ is neither associative nor commutative. Moreover, while the notation does not reflect it, the graph $G \ltimes G'$ depends on the labelings of the vertices of $G$ and $G'$. Since we are interested in overlays of commuting graphs, in this paper we will always label the vertices of commuting graphs $G$ and $G'$ in such a way that their adjacency matrices commute.

It is clear that $M(G) = G \ltimes nK_1$, $T(G) = G \ltimes G$, and $QT(G) = G \ltimes \bar{G}$, where the last two overlays are constructed by labeling the vertices of the second graph ($G$ and its complement graph $\bar{G}$, respectively) with the labels of the corresponding vertices of $G$. Note that, in each case, the adjacency matrices of the two graphs commute.

Kwak and Sato derived formulae for the Ihara zeta function and the complexity (number of spanning trees) of $L(G)$, $M(G)$, and $T(G)$, when $G$ is a regular graph [18]. In addition, Sato gave formulae for the Ihara zeta function and the complexity of $L(G)$ and $M(G)$, for semiregular graphs $G$ [23,24]. Explicit formulae for the Ihara zeta functions of other constructions (e.g. cones on regular graphs, joins of regular graphs, cylinders on such graphs) can be found in [5,6].

In this paper, we determine the characteristic polynomials of the adjacency and Laplacian matrices of an overlay of two commuting regular graphs. In addition, we give formulae for the Ihara zeta function, complexity, and Kirchhoff index of an overlay of commuting regular graphs, generalizing results from [10,18,26], and [27]. Lastly, we use our results to determine the characteristic polynomials, Ihara zeta function, complexity, and Kirchhoff index of several additional constructions, including the quasitotal graph of a regular graph and the line graph of the cone on a regular graph.

## 2. Main results

Throughout this section, $G$ and $G'$ denote simple, commuting, regular graphs of order $n$. We will assume that $G$ is connected and of regularity $r \geq 2$ (no such assumptions will be made for $G'$). Let $G$ have size $m = \frac{nr}{2}$ and $G'$ have regularity $r' \geq 0$. Label the vertices of $G$ and $G'$ such that the adjacency matrices $\mathbf{A}$ and $\mathbf{A}'$ of the graphs commute. Then the adjacency matrices $\mathbf{A}$ and $\mathbf{A}'$ can be diagonalized using the same matrix, so there is an orthonormal basis of $\mathbb{R}^n$ that consists of (common) eigenvectors $\{\mathbf{u}_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of $\mathbf{A}$ and $\mathbf{A}'$, where $\mathbf{1}_n$ denotes the vector in $\mathbb{R}^n$ with all entries equal to 1.

Let $\mathrm{Spec}_A(G) = \{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ and $\mathrm{Spec}_L(G) = \{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$ be the spectra of the adjacency and Laplacian matrices of $G$, respectively. Let $\mathrm{Spec}_A(G') = \{\lambda'_1 = r', \lambda'_2, \ldots, \lambda'_n\}$ and $\mathrm{Spec}_L(G') = \{\mu'_1 = 0, \mu'_2, \ldots, \mu'_n\}$ be the spectra of the adjacency and Laplacian matrices of $G'$, respectively (all the spectra are regarded as multisets). Since $G$ and $G'$ are regular, then $\mu_i = r - \lambda_i$ and $\mu'_i = r' - \lambda'_i$, $1 \leq i \leq n$. Throughout the paper, the eigenvalues $\lambda_i$ and $\lambda'_i$ are assumed to correspond to the same eigenvector $\mathbf{u}_i$, $1 \leq i \leq n$, of the adjacency matrices of $G$ and $G'$.

We denote by $\mathbf{B} = \mathbf{B}_G = (b_{ij})$ the edge–vertex incidence matrix of $G$, i.e. the $m \times n$ matrix with $b_{ij} = 1$ if $e_i \in E(G)$ and $v_j \in V(G)$ are incident in $G$, and $b_{ij} = 0$, otherwise. If $\mathbf{A}_{L(G)}$ denotes the adjacency matrix of the line graph of $G$, then it is well known [10] that:

$$\mathbf{A}_{L(G)} = \mathbf{B}\mathbf{B}^T - 2\mathbf{I}_m \tag{1}$$

$$\mathbf{A} = \mathbf{B}^T\mathbf{B} - r\mathbf{I}_n. \tag{2}$$

Since $G$ is $r$-regular, then the line graph $L(G)$ is $(2r-2)$-regular and the spectrum of the adjacency matrix of $L(G)$ is

$$\mathrm{Spec}_A(L(G)) = \{\bar{\lambda}_1 = 2r-2, \bar{\lambda}_2 = \lambda_2 + r - 2, \ldots, \bar{\lambda}_n = \lambda_n + r - 2, [-2]^{m-n}\}.$$

**Theorem 2.1** (*Characteristic Polynomials*)**.** *Let $G$ and $G'$ be commuting regular graphs of order $n$ and regularity $r$ and $r'$, respectively. Let $m$ be the size of $G$.*
(i) *The characteristic polynomial of the adjacency matrix of $G \ltimes G'$ is*

$$\Phi_{G\ltimes G'}(x) = (x+2)^{m-n}\prod_{i=1}^{n}[x^2 - (\lambda'_i + \bar{\lambda}_i)x + \bar{\lambda}_i(\lambda'_i - 1) - 2].$$

(ii) *The characteristic polynomial of the Laplacian matrix of $G \ltimes G'$ is*

$$\mathcal{L}_{G\ltimes G'}(x) = (x - 2r - 2)^{m-n}\prod_{i=1}^{n}[x^2 - (\mu_i + \mu'_i + r + 2)x + \mu_i(\mu'_i + r + 1) + 2\mu'_i].$$

**Proof.** (i) Note that the adjacency matrix $\mathbf{A}_{G\ltimes G'}$ of the overlay can be represented in block form

$$\mathbf{A}_{G\ltimes G'} = \begin{bmatrix} \mathbf{A}_{L(G)} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A}' \end{bmatrix}.$$