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On Induced Colourful Paths in Triangle-free Graphs

Jasine Babu [a] Manu Basavaraju [b,1] L. Sunil Chandran [c,2] Mathew C. Francis [d,3]

[a] *Indian Institute of Technology, Palakkad, India (jasinekb@gmail.com)*

[b] *National Institute of Technology Karnataka, India (manub@nitk.ac.in)*

[c] *Indian Institute of Science, Bangalore, India (sunil@csa.iisc.ernet.in)*

[d] *Indian Statistical Institute, Chennai, India (mathew@isichennai.res.in)*

Abstract

Given a graph $G = (V, E)$ whose vertices have been properly coloured, we say that a path in G is *colourful* if no two vertices in the path have the same colour. It is a corollary of the Gallai-Roy Theorem that every properly coloured graph contains a colourful path on $\chi(G)$ vertices. We explore a conjecture that states that every properly coloured triangle-free graph G contains an induced colourful path on $\chi(G)$ vertices and prove its correctness when the girth of G is at least $\chi(G)$. Recent work on this conjecture by Gyárfás and Sárközy, and Scott and Seymour has shown the existence of a function f such that if $\chi(G) \geq f(k)$, then an induced colourful path on k vertices is guaranteed to exist in any properly coloured triangle-free graph G.

Keywords: Induced Path, Colourful Path, Triangle-free Graph

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1 Introduction

All graphs considered in this paper are simple, undirected and finite. For a graph $G = (V, E)$, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. A function $c : V(G) \to \{1, 2, \ldots, k\}$ is said to be a *proper k-colouring* of G if for any edge $uv \in E(G)$, we have $c(u) \neq c(v)$. A graph is *properly coloured*, if it has an associated proper k-colouring c specified (for some k). The minimum integer k for which a graph G has a proper k-colouring is the *chromatic number* of G, denoted by $\chi(G)$. A subgraph H of a properly coloured graph G is said to be *colourful* if no two vertices of H have the same colour. If a colourful subgraph H of G is also an induced subgraph, then we say that H is an *induced* colourful subgraph of G.

It is a corollary of the classic Gallai-Roy Theorem [4] that every (not necessarily optimally) properly coloured graph G has a colourful path on $\chi(G)$ vertices (an alternative proof for this is given in Theorem 2.3). We are interested in the question of when one can find colourful paths that are also induced in a given properly coloured graph. Note that the colourful path on $\chi(G)$ vertices that should exist in any properly coloured graph G may not always be an induced path. In fact, when G is a complete graph, there is no induced path on more than two vertices in the graph. The following conjecture is due to N. R. Aravind [2].

Conjecture 1.1 *Let G be a triangle-free graph that is properly coloured. Then there is an induced colourful path on $\chi(G)$ vertices in G.*

Recently, Gyáfás and Sárközy [7] studied this conjecture and showed that there exists a function $f(k)$ such that in any properly coloured graph G with girth at least 5 and $\chi(G) \geq f(k)$, there is an induced colourful path on k vertices. This was improved by Scott and Seymour [11], who showed the existence of a function $f(k)$ such that in any properly coloured triangle-free graph G with chromatic number at least $f(k)$, there is an induced colourful path on k vertices. In fact, they proved the more general result that for any two integers k and t, there exists a function $f(k, t)$ such that in any properly coloured graph G with $\omega(G) \leq t$ and $\chi(G) \geq f(k, t)$, there is an induced colourful path on k vertices.

A necessary condition for Conjecture 1.1 to hold is the presence of an induced path on $\chi(G)$ vertices in any triangle free graph G. Indeed something stronger is known to be true: each vertex in a triangle-free graph G is the starting point of an induced path on $\chi(G)$ vertices [5]. Concerning induced trees, Gyárfás [6] and Sumner [12] conjectured that there exists an integer-valued