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Equiangular lines and subspaces in Euclidean spaces

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Abstract

A family of lines through the origin in a Euclidean space is called equiangular if any pair of lines defines the same angle. The problem of estimating the maximum cardinality of such a family in $\mathbb{R}^n$ was studied extensively for the last 70 years. Motivated by a question of Lemmens and Seidel from 1973, we prove that for every fixed angle θ and n sufficiently large, there are at most $2n-2$ lines in $\mathbb{R}^n$ with common angle θ. Moreover, this is achievable only for $\theta = \arccos \frac{1}{3}$.

We also study analogous questions for k-dimensional subspaces. We discuss natural ways of defining the angle between k-dimensional subspaces and correspondingly study the maximum size of an equiangular set of k-dimensional subspaces in $\mathbb{R}^n$, obtaining bounds which extend and improve a result of Blokhuis.

Keywords: equiangular lines, Euclidean spaces, projective spaces, equiangular subspaces, Grassmannian, principle angles

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1 Equiangular lines

A set of lines through the origin in n-dimensional Euclidean space is called *equiangular* if any pair of lines defines the same angle. Equiangular sets of lines appear naturally in various areas of mathematics. In elliptic geometry, they correspond to equilateral sets of points, i.e. regular simplexes [16]. They also appear in frame theory [17] and the theory of polytopes [8].

It is therefore a natural question to determine the maximum cardinality $N(n)$ of an equiangular set of lines in $\mathbb{R}^n$. This is also considered to be one of the founding problems of algebraic graph theory, see e.g. [15, p. 249]. While it is easy to see that $N(2) \leq 3$ and that the three diagonals of a regular hexagon achieve this bound, matters already become more difficult in 3 dimensions. This problem was first studied by Haantjes [16] in 1948, who showed that $N(3) = N(4) = 6$ and that an optimal configuration in 3 (and 4) dimensions is given by the 6 diagonals of a convex regular icosahedron. In 1966, van Lint and Seidel [22] formally posed the problem of determining $N(n)$ for all positive integers n and furthermore showed that $N(5) = 10$, $N(6) = 16$ and $N(7) \geq 28$.

A general upper bound of $\binom{n+1}{2}$ on $N(n)$ was established by Gerzon (see [20]). Let us outline his proof. Given an equiangular set of m lines in $\mathbb{R}^n$, one can choose a unit vector x_i along the ith line to obtain vectors $x_1, \dots, x_m$ satisfying $\langle x_i, x_j \rangle \in \{-\alpha, \alpha\}$ for $i \neq j$. Consider the family of outer products $x_i x_i^\intercal$; they live in the $\binom{n+1}{2}$-dimensional space of symmetric $n \times n$ matrices, equipped with the inner product $\langle A, B \rangle = \mathrm{tr}(A^\intercal B)$. It is a routine calculation to verify that $\left\langle x_i x_i^\intercal, x_j x_j^\intercal \right\rangle = \langle x_i, x_j \rangle^2$, which equals α^2 if $i \neq j$ and 1 otherwise. This family of matrices is therefore linearly independent, which implies $m \leq \binom{n+1}{2}$.

In dimensions 2 and 3 this gives upper bounds of 3 and 6, respectively, matching the actual maxima. In $\mathbb{R}^7$, the above bound shows $N(7) \leq 28$. This can be achieved by considering the set of all 28 permutations of the vector $(1, 1, 1, 1, 1, 1, -3, -3)$, see [22,24]. It is also known that there is an equiangular set of 276 lines in $\mathbb{R}^{23}$, see e.g. [20], which again matches Gerzon's bound. Strikingly, these four examples are the only known ones to match his bound [3]. In fact, for a long time it was even an open problem to determine whether n^2 is the correct order of magnitude. In 2000, de Caen [6] constructed a set of $2(n+1)^2/9$ equiangular lines in $\mathbb{R}^n$ for all n of the form $3 \cdot 2^{2t-1} - 1$. Subsequently, several other constructions of the same order were found [3,14,18]. For further progress on finding upper and lower bounds on $N(n)$ see e.g. [3] and its references.