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Staircases, dominoes, and the growth rate of 1324-avoiders

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Abstract

We establish a lower bound of 10.271 for the growth rate of the permutations avoiding 1324, and an upper bound of 13.5. This is done by first finding the precise growth rate of a subclass whose enumeration is related to West-2-stack-sortable permutations, and then combining copies of this subclass in particular ways.

Keywords: Permutation, patterns, enumeration, growth rate.

1 Introduction

The class of 1324-avoiding permutations is famously hard to count. Whereas every other permutation class that avoids a single length 4 permutation was

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enumerated in the 1990s (see Bóna [5] and Gessel [14]), not even the first-order asymptotics (the "growth rate") of Av(1324) is yet known.

Let π and σ be permutations of lengths n and m respectively, written in one-line notation. We say that σ is *contained* in π if there exists a subsequence $i_1 < i_2 < \cdots < i_m$ of $1, \ldots, n$ such that $\sigma(j) < \sigma(k)$ if and only if $\pi(i_j) < \pi(i_k)$, for all $1 \leq j, k \leq m$. If σ is not contained in π, then it *avoids* π. We write $\mathrm{Av}(\pi)$ to mean the set consisting of all permutations that avoid π, and note that it forms a hereditary class, or *permutation class*, in the sense that whenever $\sigma \in \mathrm{Av}(\pi)$ and τ is contained in σ, then $\tau \in \mathrm{Av}(\pi)$.

Given any permutation π, let $S_n(\pi)$ denote the number of permutations of length n that avoid π. The *growth rate* of the class $\mathrm{Av}(\pi)$ is

$$\mathrm{gr}(\mathrm{Av}(\pi)) = \lim_{n\to\infty} \sqrt[n]{S_n(\pi)},$$

and is known to exist by a result of Arratia [3], combined with the celebrated resolution of the Stanley-Wilf conjecture by Marcus and Tardos [15].[3] More generally, for an infinite sequence $s_1, s_2, \ldots$ of positive integers, the *growth rate* of (s_n) is $\lim_{n\to\infty} \sqrt[n]{s_n}$, if this exists.

In the same paper, Arratia [3] conjectured that $\mathrm{gr}(\mathrm{Av}(\pi)) \leq (|\pi| - 1)^2$, where $|\pi|$ denotes the length of π. However this conjecture was refuted in 2006 by Albert, Elder, Rechnitzer, Westcott and Zabrocki [1], by proving that $\mathrm{gr}(\mathrm{Av}(1324)) \geq 9.47$, thereby cementing Av(1324) as the *bête noire* of permutation classes. Indeed, during a conference in 2004 when the result of [1] was announced, Doron Zeilberger famously declared that "not even God knows $S_{1000}(1324)$". Humans, with the help of computers, now know $S_{36}(1324)$, and Conway and Guttman's analysis [13] of their computation provides an estimate for $\mathrm{gr}(\mathrm{Av}(1324))$ of 11.60 ± 0.01, and they conjecture that $S_n(1324) \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$ where $\sigma = \frac{1}{2}$, which would imply that this sequence does not have an algebraic singularity.

The history of rigorous lower and upper bounds for $\mathrm{gr}(\mathrm{Av}(1324))$ now spans several papers, and is summarised in Table 1. In addition to these, Claesson, Jelínek and Steingrímsson [12] make a conjecture regarding the number of permutations with a fixed number of inversions of each length, which if resolved would give an improved upper bound of $e^{\pi\sqrt{2/3}} \approx 13.001954$.

Our contribution to the growth rate study of Av(1324) is to provide new

[3] The existence of growth rates for general permutation classes (i.e. those avoiding one or more permutations) remains open: Marcus and Tardos [15] only guarantees that lim sup exists.