Available online at www.sciencedirect.com

ScienceDirect

Electronic Notes in DISCRETE MATHEMATICS

Electronic Notes in Discrete Mathematics 61 (2017) 147–153

www.elsevier.com/locate/endm

Decomposition of 8-regular graphs into paths of length 4

Fábio Botler [1,2]

Departamento de Ingeniería Industrial
Universidad de Chile
Santiago, Chile

Alexandre Talon [3]

LIP
École Normale Supérieure de Lyon
Lyon, France

Abstract

A P_ℓ-decomposition of a graph G is a set of edge-disjoint copies of P_ℓ in G that cover the edge set of G, where P_ℓ is the path with ℓ edges. Kouider and Lonc [M. Kouider, Z. Lonc, Path decompositions and perfect path double covers, Australas. J. Combin. 19 (1999) 261–274] conjectured that any 2ℓ-regular graph G admits a P_ℓ-decomposition $\mathcal{D}$ where every vertex of G is the end-vertex of exactly two paths of $\mathcal{D}$. In this paper we verify Kouider and Lonc's Conjecture for paths of length 4.

Keywords: Graph decomposition, regular graph, path

[1] F. Botler is partially supported by Millenium Nucleus Information and Coordination in Networks (ICM/FIC RC 130003), and FONDECYT (project N^o3170878).

[2] Email: `fbotler@ime.usp.br`

[3] Email: `alexandre.talon@ens-lyon.fr`

http://dx.doi.org/10.1016/j.endm.2017.06.032

1 Introduction

A decomposition of a graph G is a set $\mathcal{D}$ of subgraphs of G that partitions the edge set of G. Given a graph H, we say that $\mathcal{D}$ is an H-decomposition of G if every element of $\mathcal{D}$ is isomorphic to H. Ringel (1963) conjectured that the complete graph $K_{2\ell+1}$ admits a T-decomposition for any tree T with ℓ edges. Ringel's Conjecture holds for many classes of trees such as stars, paths, and bistars (see [2,6]). Häggkvist [3] generalized Ringel's Conjecture as follows.

Conjecture 1.1 (Graham–Häggkvist, 1989) *Let T be a tree with ℓ edges. If G is a 2ℓ-regular graph, then G admits a T-decomposition*

Häggkvist [3] also proved Conjecture 1.1 when G has girth at least the diameter of T. For the case where $T = P_\ell$ is the path with ℓ edges (note that this notation is not standard), Kouider and Lonc [4] improved Häggkvist's result proving that *if G is a 2ℓ-regular graph with girth $g \geq (\ell+3)/2$, then G admits a* balanced *P_ℓ-decomposition $\mathcal{D}$*, that is a path decomposition $\mathcal{D}$ where each vertex is the end-vertex of exactly two paths of $\mathcal{D}$. These authors also stated the following strengthening of Conjecture 1.1 for paths.

Conjecture 1.2 (Kouider–Lonc, 1999) *Let ℓ be a positive integer. If G is a 2ℓ-regular graph, then G admits a balanced P_ℓ-decomposition.*

One of the authors [1] proved the following weakening of Conjecture 1.2: for every positive integer ℓ, there exists an integer $m_0 = m_0(\ell)$ such that, if G is a $2m\ell$-regular graph with $m \geq m_0$, then G admits a P_ℓ-decomposition $\mathcal{D}$ such that every vertex of G is the end-vertex of exactly $2m$ paths of $\mathcal{D}$. In this paper we prove Conjecture 1.2 in the case $\ell = 4$.

Notation. A *trail* T is a graph for which there is a sequence $B = x_0 \cdots x_\ell$ of its vertices such that $E(T) = \{x_i x_{i+1} \colon 0 \leq i \leq \ell - 1\}$ and $x_i x_{i+1} \neq x_j x_{j+1}$, for every $i \neq j$. Such a sequence B of vertices is called a *tracking* of T. Given a tracking $B = x_0 \cdots x_\ell$ we denote by B^- the tracking $x_\ell \cdots x_0$. We denote by $V(B)$ and $E(B)$ the sets $\{x_0, \ldots, x_\ell\}$ of vertices, and $\{x_i x_{i+1} \colon 0 \leq i \leq \ell-1\}$ of edges of B, respectively. Moreover, we denote by $\bar{B}$ the trail $\big(V(B), E(B)\big)$, and by *length* of B we mean the length of $\bar{B}$. We also use *ℓ-tracking* to denote a tracking of length ℓ. A set of edge-disjoint trackings $\mathcal{B}$ of a graph G is a *tracking decomposition* of G if $\cup_{B \in \mathcal{B}} E(B) = E(G)$, and if every tracking of $\mathcal{B}$ induces a path, we say that $\mathcal{B}$ is a *path tracking decomposition.*

Suppose that every tracking in $\mathcal{B}$ has length at least 2 and consider an orientation O of a set of edges of G as follows. For each tracking $B = x_0 \cdots x_\ell$ in $\mathcal{B}$, orient $x_0 x_1$ from x_1 to x_0, and $x_{\ell-1} x_\ell$ from $x_{\ell-1}$ to x_ℓ. Given a vertex v of G, we denote by $\mathcal{B}(v)$ (resp. $\mathrm{Hang}(v, \mathcal{B})$) the number of edges of G directed towards (resp. leaving) v in O (i.e., $\mathcal{B}(v) = d^-_O(v)$ and $\mathrm{Hang}(v, \mathcal{B}) = d^+_O(v)$). We say that an edge that leaves v in O is a *hanging* edge at v, and that a