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K_4-expansions have the edge-Erdős-Pósa property

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Abstract

A family $\mathcal{F}$ of graphs has the edge-Erdős-Pósa property if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \in \mathbb{N}$ and every graph G, the following holds: either G contains k edge-disjoint subgraphs contained in $\mathcal{F}$ or there is an edge set Y of size at most $f(k)$ such that $G - Y$ does not contain any subgraph in $\mathcal{F}$. We prove that the family of graphs that have K_4 as a minor has the edge-Erdős-Pósa property.

Keywords: Packing, Covering, Minor, Erdős and Pósa, vertex version, edge version

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1 Background

Packing and covering of cycles in graphs are two tightly related problems. In 1965, Erdős and Pósa [6] proved that for every graph G and every integer k, there are either k disjoint cycles in G (packing) or a set $X \subseteq V(G)$ of size $O(k \log k)$ such that $G - X$ is acyclic (covering). The set X is called *hitting set*.

Inspired by this classic theorem, many authors investigated which other objects enjoy such an *Erdős-Pósa property*: that there are always k disjoint such objects or a vertex set that hits every target object and whose size is bounded by a function of k. This is the case, for example, for long cycles [1,7,12], cycles through prescribed vertices [11,13,3] and for cycles with parity or modularity constraints [9,10,17,16]. The Erdős-Pósa property does not only hold for different types of cycles but also for many more families of graphs. The celebrated theorem of Robertson and Seymour establishes the Erdős-Pósa property for a whole host of graph classes in one fell swoop:

Theorem 1.1 (Robertson and Seymour [15]) *The family of expansions of a fixed graph H has the Erdős-Pósa property if and only if H is planar.*

(A graph X is an *H-expansion* if H is a minor of X.)

1.1 Vertex version vs. edge version

The Erdős-Pósa property asks for vertex-disjoint objects or a vertex hitting set. It makes as much sense to ask for edge-disjoint objects or a hitting set of edges. Indeed, the original Erdős-Pósa theorem has an edge-analogue: for every $k \in \mathbb{N}$ and every graph G, there are either k *edge-disjoint* cycles in G or an *edge set* Y of size $O(k \log k)$ such that $G - Y$ is acyclic. (This is an exercise in [4].)

Most of the research in this area focuses on the vertex version, and far less on the edge version. Families of objects that are known to enjoy the edge-Erdős-Pósa property are: cycles through prescribed vertices [13], θ_r-graphs (r internally disjoint paths between two vertices, [14]), immersions of a fixed planar subcubic graph H [8] and long cycles [2].

All these families were known before to have the vertex-Erdős-Pósa property. Does Robertson and Seymour's theorem extend to an edge version, too?

Conjecture 1.2 *The family of H-expansions has the edge-Erdős-Pósa property if and only if H is planar.*