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Directed Ramsey number for trees

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Abstract

Given an oriented graph H, the k-colour *oriented Ramsey number* of H, denoted by $\vec{r}(H,k)$, is the least integer n, for which every k-edge-coloured tournament on n vertices contains a monochromatic copy of H. We show that $\vec{r}(T,k) \leq c_k|T|^k$ for any oriented tree T, which, in general, is tight up to a constant factor. We also obtain a stronger bound, when H is an arbitrarily oriented path.

Keywords: Ramsey theory, directed graph, directed tree, tournament.

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1 Introduction

An *oriented graph* is a directed graph in which between any two vertices there is at most one edge. The *underlying graph* of a directed graph is the graph obtained by removing orientation from its edges.

One of the most classical results in the theory of directed graphs is the Gallai-Hasse-Roy-Vitaver theorem [8, 11, 15, 16], abbreviated as the GHRV theorem, which states that any directed graph, whose underlying graph has chromatic number at least n, contains a directed path of length $n-1$. Note that, by the length of a path we mean the number of *edges* in the path.

It is natural to ask if there are directed graphs, other than the directed path, which are guaranteed to exist in any n-chromatic oriented graph. We note that such a result is not true for any graph containing a directed cycle as we can, given any n-chromatic graph, orient the edges according to an order of the vertices, thus obtaining an acyclic n-chromatic graph.

This question was first asked by Burr [4] in 1980, who conjectured that any $(2n-2)$-chromatic digraph contains any oriented tree of order n. If true, this is best possible, as a regular tournament on $2n-3$ vertices is clearly $(2n-3)$-chromatic, but has maximum out-degree $n-2$, so it does not contain an out-directed star on n vertices. The conjecture is still widely open and even its weakening, where $2n-2$ is replaced by cn for a large constant c, is not known. Burr showed that the statement holds for $(n-1)^2$-chromatic digraphs, and the best general result in this direction is due to Addario-Berry, Havet, Linhares Sales, Reed and Thomassé [1] who proved it for $(n^2/2-n/2+1)$-chromatic digraphs. It is of note that the conjecture is open even for relatively simple trees, such as arbitrarily oriented paths (though the very special case of directed paths with two blocks is solved [2, 6]).

The GHRV theorem has an interesting application to Ramsey theory. It implies that any 2-edge-colouring of a tournament on n^2+1 vertices contains a monochromatic path of length n. Indeed, given such a tournament T, we apply the theorem to the subgraph T_R consisting of the red edges in T. Then either T_R contains a directed path of length n, in which case we are done, or the chromatic number of T_R is at most n, so there is an independent set of size at least $n+1$. This translates into the existence of a subtournament of T of order $n+1$ consisting only of blue edges. Since every tournament has a Hamiltonian path, we find a blue path of length n.

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