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Monochromatic paths in random tournaments

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Abstract

We prove that, with high probability, any 2-edge colouring of a random tournament on n vertices contains a monochromatic path of length $\Omega(n/\sqrt{\log n})$. This resolves a conjecture of Ben-Eliezer, Krivelevich and Sudakov and implies a nearly tight upper bound on the oriented size Ramsey number of a directed path.

Keywords: Random tournament, Size Ramsey number, edge colouring, directed paths.

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1 Introduction

Ramsey theory refers to a large body of mathematical results, which roughly say that any sufficiently large structure is guaranteed to have a large well-organised substructure. For example, the celebrated theorem of Ramsey [7] says that for any fixed graph H, every 2-edge-colouring of a sufficiently large complete graph contains a monochromatic copy of H. The *Ramsey number of H* is defined to be the smallest order of a complete graph satisfying this property.

In this work, we study an analogous phenomenon for oriented graphs. An *oriented graph* is a directed graph which can be obtained from a simple undirected graph by orienting its edges. A *tournament* is an orientation of the complete graph. Given directed graphs G and H, we write $G \to H$ if any 2-edge-colouring of G contains a monochromatic copy of H. The *oriented Ramsey number* of H is defined to be the smallest n for which $T \to H$ for every tournament T on n vertices.

While in the undirected case the Ramsey number exists and is finite for every H, in the directed case the Ramsey number can be infinite. Indeed, given a directed graph H that contains a directed cycle and any tournament T, we colour the edges of T as follows. Denote the vertices of T by $v_1, \ldots, v_n$ and colour edges $v_i v_j$ red if $i < j$ and blue otherwise. It is easy to see that this coloring has no monochromatic directed cycle, so, in particular, there is no monochromatic copy of H.

Therefore it only makes sense to study the Ramsey properties for acyclic graphs. We note that oriented Ramsey number of an acyclic graph is always finite. This follows since every acyclic graph is subgraph of a transitive tournament, and transitive tournaments have finite oriented Ramsey number by essentially the same argument as for the complete graphs in the undirected Ramsey case. One of the most basic acyclic graphs is a directed path. In this paper we study the problem of finding such monochromatic paths in tournaments.

Another major difference between the undirected and the oriented Ramsey numbers, is the fact that there is only one complete graph on n vertices, while there are many tournaments on n vertices. In particular, the answer to how long a monochromatic path we can find in a tournament on n vertices T, depends on T as well as the colouring of the edges. We thus define $l(T)$ to be

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