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Stability of extremal hypergraphs with applications to an edge-coloring problem

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Abstract

We obtain sharper stability results for hypergraphs that avoid a copy of an expanded complete 2-graph and for hypergraphs that avoid a Fan subhypergraph. We apply this to an edge-coloring problem on uniform hypergraphs, where we wish to find an n-vertex hypergraph with the largest number of colorings avoiding a rainbow copy of a fixed hypergraph F.

Keywords: extremal graph theory, hypergraphs, stability, rainbow colorings

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1 Stability of extremal hypergraphs

This paper deals with extremal problems and stability in uniform hypergraphs. As usual, for an integer $r \geq 2$, an r-uniform hypergraph $H = (V, E)$ is a pair consisting of a finite *vertex set* V and of a set $E \subseteq \binom{V}{r}$ of *hyperedges*, where $\binom{V}{r}$ is the set of all subsets of V with cardinality r. As described in a seminal paper of Katona [7], the archetypal problem is to fix a set $\mathcal{C}$ of conditions on the *edge set* E and, for fixed $r \geq 2$, characterize the n-vertex r-uniform hypergraphs $H = (V, E)$ with the property that E satisfies $\mathcal{C}$ and $|E|$ is maximum. These are called *extremal hypergraphs* for this problem. For instance, in the case where $\mathcal{C} = \mathcal{C}_\mathcal{F}$ is the property that the edge set E does not contain a copy of any member of a fixed family $\mathcal{F}$ of r-uniform hypergraphs, this is the well-known *hypergraph Turán problem* for $\mathcal{F}$. Using standard terminology, we let $\text{Forb}_\mathcal{F}(n)$ be the family of n-vertex $\mathcal{F}$*-free* hypergraphs, that is, the family of all labeled r-uniform hypergraphs on n vertices that do not contain a copy of any $F \in \mathcal{F}$. The *Turán number* for $\mathcal{F}$ is the quantity $\text{ex}(n, \mathcal{F}) = \max\{|E(H)|\colon H \in \text{Forb}_\mathcal{F}(n)\}$ and the hypergraphs $H = (V, E) \in \text{Forb}_\mathcal{F}(n)$ such that $|E| = \text{ex}(n, F)$ are called $\mathcal{F}$*-extremal*. When $\mathcal{F} = \{F\}$, we omit the brackets for simplicity.

Many such general problems have the property that the extremal hypergraph is unique up to isomorphism, and that any configuration $H = (V, E)$ such that $|E|$ is close to the maximum size must be structurally close to the extremal configuration. This is known as *stability*, a concept that was introduced in the graph-theoretical setting by Erdős and Simonovits [16] and that appears naturally in several instances of the hypergraph Turán problem. We shall provide precise statements for particular hypergraph families that will be the focus of our work. We should also mention that, unlike the graph case, there are few general results about the Turán problem for hypergraphs. For more information about this problem and stability results for hypergraphs, we refer the reader to surveys by Keevash [8] and by Mubayi and Verstraëte [14].

For integers $\ell \geq r \geq 2$, let $\mathcal{K}_{\ell+1}^{(r)}$ be the family of r-uniform hypergraphs $H = (V, E)$ with at most $\binom{\ell+1}{2}$ hyperedges that contain an $(\ell+1)$-set $A \subseteq V$, called the *core* of H, with the property that, for every pair $h \in \binom{A}{2}$, there is a hyperedge in E containing h. One particular element of this family is the *expanded complete 2-graph* $H_{\ell+1}^{(r)}$, which is obtained from the complete graph $K_{\ell+1}$ on $(\ell+1)$ vertices as follows. Its core A is given by the vertex set of $K_{\ell+1}$ and every edge of $K_{\ell+1}$ is enlarged by $(r-2)$ new vertices to become an edge of $H_{\ell+1}^{(r)}$ (the sets of new vertices associated with distinct edges are pairwise