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# Threshold functions for small subgraphs: an analytic approach

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**Abstract**

We revisit the problem of counting the number of copies of a fixed graph in a random graph or multigraph, including the case of constrained degrees. Our approach relies heavily on analytic combinatorics and on the notion of *patchwork* to describe the possible overlapping of copies.

*Keywords:* random graphs, subgraphs, analytic combinatorics, generating functions.

## 1 Introduction

Since the introduction of the random graph models $G(n, m)$ and $G(n, p)$ by Erdős and Rényi [8] in 1960, one of the most studied parameters is the number $X_F$ of subgraphs isomorphic to a given graph $F$. By the asymptotic equivalence between $G(n, p)$ and $G(n, m)$, results from one model can be rigorously translated into the other one. Erdős and Rényi derived the threshold for $\{X_F > 0\}$ when $F$ is a *strictly balanced* graph (see definition next page), and Bollobás [3] generalized their result to any graph $F$. Ruciński [15] proved that $X_F$ is asymptotically normal beyond the threshold, and follows a Poisson law at the threshold iff $F$ is strictly balanced. Then Janson, Oleszkiewicz and Ruciński [12] developed a moment-based method for estimating $\mathbb{P}(X_F \geq (1+\varepsilon)\mathbb{E}(X_F))$. The notion of *strongly balanced graphs*, introduced by Ruciński and Vince in [16], plays a key role in obtaining the results mentioned above.

Recently, there has been an increasing interest in the study of constrained random graphs, such as given degree sequences or regular graphs; the number of given subgraphs in such structures has been also studied. E.g., Wormald [18] proved that the number of short cycles in these structures asymptotically follows a Poisson distribution; using a multi-dimensional saddle-point approach,

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