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Tight bounds on the coefficients of partition functions via stability

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Abstract

We show how to use the recently-developed occupancy method and stability results that follow easily from the method to obtain extremal bounds on the individual coefficients of the partition functions over various classes of bounded degree graphs.

As applications, we prove new bounds on the number of independent sets and matchings of a given size in regular graphs. For large enough graphs and almost all sizes, the bounds are tight and confirm the Upper Matching Conjecture of Friedland, Krop, and Markström, and a conjecture of Kahn on independent sets for a wide range of parameters. Additionally we prove tight bounds on the number of q-colorings of cubic graphs with a given number of monochromatic edges, and tight bounds on the number of independent sets of a given size in cubic graphs of girth at least 5.

Keywords: Partition function, stability, Upper Matching Conjecture

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1 Introduction

A Gibbs measure is a probability distribution on states of a system in which each state possesses an energy, and a state occurs with probability proportional to an exponential in its energy. The normalising constant, called the *partition function*, has a wide range of important properties. Of particular interest here are the facts that expectations over the Gibbs measure can be computed from the partition function, and that the coefficients of the partition function correspond to the numbers of states with fixed energy.

Independent sets, matchings, and colourings are fundamental objects of study in graph theory. The fact that a many different phenomena can be represented in graphs means that bounds on the number of these objects appear throughout mathematics; such as a question of Granville on which d-regular graph on n vertices has the most independent sets (see [1]). By considering appropriate Gibbs measures, some of these questions can be framed in terms of bounds on the partition function of models from statistical physics. The hard-core model and monomer-dimer model correspond to independent sets and matchings respectively. We also consider the Potts model and how it relates to colourings of graphs.

Stability is a wide-ranging theme in combinatorics which concerns whether structures that nearly attain an extreme value of some parameter must closely resemble one another. The archetypal stability result in graph theory is the theorem of Erdős and Simonovits [10,15] which states that K_{r+1}-free graphs with close to the maximum possible number of edges must closely resemble a complete, balanced r-partite graph.

In this extended abstract we discuss how stability results follow easily from the recent occupancy method [6], which gives bounds on the partition function in a variety of Gibbs measures over certain classes of graphs [4,6,7,13]. As an application we derive bounds on the individual coefficients of the partition functions. In particular, for large enough graphs and almost all sizes, we give tight upper bounds on the number of independent sets and matchings of fixed size in regular graphs, confirming conjectures of Kahn [12] and Friedland, Krop, and Markström [11] for a wide range of parameters. We give details in a forthcoming paper [8], here choosing to focus on the example of matchings in regular graphs.

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