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The evolution of random graphs on surfaces

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Abstract

For integers $g, m \geq 0$ and $n > 0$, let $S_g(n, m)$ denote the graph taken uniformly at random from the set of all graphs on $\{1, 2, \ldots, n\}$ with exactly $m = m(n)$ edges and with genus at most g. We use counting arguments to investigate the components, subgraphs, maximum degree, and largest face size of $S_g(n, m)$, finding that there is often different asymptotic behaviour depending on the ratio $\frac{m}{n}$.

Keywords: random graphs, surfaces, components, subgraphs, maximum degree, largest face.

1 Introduction

1.1 Background and motivation

Random planar graphs have been the subject of much activity, and many properties of the standard random planar graph $P(n)$ (taken uniformly at random from the set of all planar graphs with vertex set $[n] := \{1, 2, \ldots, n\}$) are now known. For example, asymptotic results have been obtained for the

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probability that $P(n)$ will contain given components and subgraphs [15,20], for the number of vertices of given degree [10], and for the size of the maximum degree and largest face [9,19]. In addition, clever algorithms for generating and sampling planar graphs have been designed [3,11], and random planar maps have been studied [12,13].

The classical Erdős-Rényi random graph $G(n, m)$ is taken uniformly at random from the set of all graphs on $[n]$ with exactly m edges. Hence, it is natural to also examine the planar analogue $P(n, m)$, taken uniformly at random from the set of all planar graphs on $[n]$ with exactly m edges. Note that the extra condition on the number of edges typically makes $P(n, m)$ more challenging to study than $P(n)$, but it also has the effect of producing richer and more complex behaviour, and many exciting results have been obtained [1,2,5,6,7,14,15,17].

It is well known that $P(n, m)$ behaves in the same way as $G(n, m)$ if $m < \frac{n}{2} - \omega(n^{2/3})$, since the probability that $G(n, m)$ will be planar converges to 1 as $n \to \infty$ for this range of m (see, for example, [16]). However, different properties have been found to emerge when we are beyond this region [6,7,14,15,17].

In this work, we are interested in graphs with genus at most g. A graph is said to have genus at most g if it can be embedded without any crossing edges on an orientable surface of genus g (i.e. a sphere to which g handles have been attached). Hence, the simplest case when $g = 0$ corresponds to planar graphs.

We let $S_g(n)$ denote the graph taken uniformly at random from the set of all graphs on $[n]$ with genus at most g, and we let $S_g(n, m)$ denote the graph taken uniformly at random from the set $\mathcal{S}^g(n, m)$ of all graphs on $[n]$ with exactly m edges and with genus at most g (it is known that this then implies that we must have $m \leq 3n - 6 + 6g$). Throughout this work, $m = m(n)$ will be a function of n, while g will be a constant independent of n.

Many of the results on $P(n)$ (i.e. $S_0(n)$) have now also been generalised to $S_g(n)$ [4,18,19]. However, similar extensions have not yet been achieved for the full $S_g(n, m)$ case, and so that is to be the subject of this work.

We investigate the probability that $S_g(n, m)$ will contain given components and subgraphs, as well as the size of the maximum degree and largest face (maximised over all embeddings with genus at most g). We find that the restriction on the number of edges enriches the results, by providing different behaviour depending on the ratio $\frac{m}{n}$. Hence, this change as $\frac{m}{n}$ varies can be thought of as the 'evolution' of random graphs on surfaces.