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# Exponentially many nowhere-zero $\mathbb{Z}_3$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-flows

Zdeněk Dvořák [1,4,a] Bojan Mohar [2,5,b] Robert Šámal [3,6,a]

[a] *CSI, Faculty of Mathematics and Physics*
*Charles University*
*Prague, Czech Republic*

[b] *Department of Mathematics*
*Simon Fraser University*
*Burnaby, Canada*

**Abstract**

We prove that, in several settings, a graph has exponentially many nowhere-zero flows. Our results may be seen as a counting alternative to the well-known proofs of existence of $\mathbb{Z}_3$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-flows. In the dual setting, proving exponential number of 3-colorings of planar triangle-free graphs is a related open question due to Thomassen.

*Keywords:* graphs, nowhere-zero flows, counting

[1] Email: `rakdver@iuuk.mff.cuni.cz`
[2] Email: `mohar@sfu.ca`
[3] Email: `samal@iuuk.mff.cuni.cz`

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# 1 Introduction

Our graphs may have loops and parallel edges. Let $G$ be a graph with a given orientation of its edges and let $\Gamma$ be an abelian group. A mapping $\varphi : E(G) \to \Gamma$ is called a *flow* if for every vertex $v \in V(G)$ it satisfies the Kirchhoff's law

$$\sum_{e \in \delta^+(v} \varphi(e) = \sum_{e \in \delta^-(v} \varphi(e)$$

(here $\delta^+(v)$, $\delta^-(v)$ denote the set of edges directed away from/toward $v$). We say that $\varphi$ is a $\Gamma$-flow to express the group we are using. Further, we say that $\varphi$ is *nowhere-zero* if $\varphi(e) \neq 0$ for every $e \in E(G)$ and we say that $\varphi$ is a $k$-flow if $\Gamma = \mathbb{Z}$ and $|\varphi(e)| < k$ for every edge $e$. Note that, while we need some orientation to define a flow, the orientation itself is irrelevant: if we reverse an arc and change the sign of its flow-value, the Kirchhoff's law still holds true.

The study of nowhere-zero flows was initiated by Tutte. The main motivation was the following duality theorem:

**Theorem 1.1** *Let $G$ be a plane graph and let $G^*$ be the dual of $G$. Then $G$ has a nowhere-zero $\mathbb{Z}_k$-flow if and only if $G^*$ is $k$-colorable. If $f_k$ is the number of nowhere-zero $\mathbb{Z}_k$-flows on $G$ and $c_k$ the number of $k$-colorings of $G^*$, then $c_k = k \cdot f_k$.*

As a consequence, the questions about chromatic number of planar graphs, that were always at the core of graph theory, can be studied in a new setting. This line of thought had lead to the following conjectures, motivated by the Grötzsch's theorem and by the Four Color Theorem (still a conjecture then).

**Conjecture 1.2 (Tutte)**

- *Every 4-edge-connected graph has a $\mathbb{Z}_3$-flow.*
- *Every 2-edge-connected graph with no Petersen minor has a $\mathbb{Z}_4$-flow.*
- *Every 2-edge-connected graph has a $\mathbb{Z}_5$-flow.*

Tutte proved that, surprisingly, the concept of $\Gamma$-flows does not depend on the structure of $\Gamma$, but only on its size. Tutte also proved that the number of nowhere-zero $\Gamma$-flows on $G$ is equal to $p_G(|\Gamma|)$ for some polynomial $p_G$ depending on $G$. Thus, if the graph is fixed and we enlarge the group, the number of nowhere-zero flows grows polynomially. In this paper, we will show that when we do the opposite—keep the same group and enlarge the graph—then in many cases the number of nowhere-zero flows grows exponentially.

Tutte's 3-flow, 4-flow, and 5-flow conjectures (Conjecture 1.2) are still