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Irreducible 4-critical triangle-free toroidal graphs

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Abstract

The theory of Dvořák, Král', and Thomas [5] shows that a 4-critical triangle-free graph embedded in the torus has only a bounded number of faces of length greater than 4 and that the size of these faces is also bounded. We study the natural reduction in such embedded graphs—identification of opposite vertices in 4-faces. We give a computer-assisted argument showing that there are exactly four 4-critical triangle-free *irreducible* toroidal graphs in which this reduction cannot be applied without creating a triangle. Using this result, we show that every 4-critical triangle-free graph embedded in the torus has at most four 5-faces, or a 6-face and two 5-faces, or a 7-face and a 5-face, in addition to at least seven 4-faces. This result serves as a basis for the exact description of 4-critical triangle-free toroidal graphs, which we present in a followup paper.

Keywords: graph coloring, toroidal graphs, triangle-free graphs

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1 Introduction

The subject of coloring graphs on surfaces goes back to the work of Heawood [12], who proved that any graph G drawn in surface Σ is t-colorable for any t satisfying $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24\gamma_\Sigma} + 1)/2 \rfloor$ unless Σ is the sphere (where γ_Σ denotes the Euler genus of Σ defined as $\gamma = 2k$ for the sphere with k handles and $\gamma = k$ for the sphere with k crosscaps). Incidentally, the assertion holds for the sphere as well, as stated by the Four-Color Theorem [2,3,16]. The bound given by Heawood's formula is tight—as proven by Ringel and Youngs [15], the bound is the best possible for all surfaces except the Klein bottle, for which the correct bound is 6.

While Heawood's formula gives a tight bound on the possible values of the chromatic number of graphs on almost all surfaces, values close to the bound are achieved by only relatively few graphs. Thomassen [20] proved that for every $k \geq 6$ and each surface Σ, there are only finitely many k-critical graphs that can be drawn in Σ (a graph is *k-critical* if it is not $(k-1)$-colorable, but all its proper subgraphs are $(k-1)$-colorable). An immediate consequence from a computational point of view is that for every fixed surface Σ and a graph G drawn in Σ, it is possible to efficiently test whether G is $(k-1)$-colorable by testing the presence of all possible k-critical subgraphs. The lists of all k-critical graphs for $k \geq 6$ are explicitly known for the projective plane [1], the torus [18] and the Klein bottle [13,4].

Since 2-colorability is polynomial-time solvable and 3-colorability is NP-complete even for planar graphs [10], the only remaining non-trivial case is 4-colorability. It is an open problem whether there is a polynomial-time algorithm for testing 4-colorability of graphs in a fixed surface other than the sphere. However, a characterization similar to the one described for 5-colorability above does not exists, as shown by an elegant construction of infinitely many 5-critical graphs in any such surface by Fisk [9].

We consider the analogous problem for embedded graphs of larger girth. Thomassen [21] proved that for every $k \geq 4$ and each surface Σ, there are only finitely many k-critical graphs of girth at least five that can be drawn in Σ. Consequently, testing the $(k-1)$-colorability of graphs with girth at least five again reduces to deciding the presence of finitely many obstructions. There actually turn out to be no 4-critical graphs of girth at least five in the projective plane and the torus [19], and in the Klein bottle [17], i.e., all such graphs are 3-colorable.

It is also easy to see that for every $k \geq 5$, there are only finitely many k-critical triangle-free graphs that can be drawn in Σ. However, for $k = 4$