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Burling graphs, chromatic number, and orthogonal tree-decompositions [1]

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Abstract

A classic result of Asplund and Grünbaum states that intersection graphs of axis-aligned rectangles in the plane are χ-bounded. This theorem can be equivalently stated in terms of path-decompositions as follows: There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph that has two path-decompositions such that each bag of the first decomposition intersects each bag of the second in at most k vertices has chromatic number at most $f(k)$. Recently, Dujmović, Joret, Morin, Norin, and Wood asked whether this remains true more generally for two tree-decompositions. In this note we provide a negative answer: There are graphs with arbitrarily large chromatic number for which one can find two tree-decompositions such that each bag of the first decomposition intersects each bag of the second in at most two vertices. Furthermore, this remains true even if one of the two decompositions is restricted to be a path-decomposition. This is shown using a construction of

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triangle-free graphs with unbounded chromatic number due to Burling, which we believe should be more widely known.

Keywords: chromatic number, tree decompositions, path decompositions

1 Burling graphs

For each $k \geqslant 1$, we define the *Burling graph* G_k and a collection $\mathcal{S}(G_k)$ of stable sets of G_k by induction on k as follows. First, let G_1 be the graph consisting of a single vertex and let $\mathcal{S}(G_1)$ contain just the single vertex stable set of G_1. Next, suppose $k \geqslant 2$ for the inductive case. First, take a copy H of G_{k-1}, which we think of as the 'master' copy. For each stable set $S \in \mathcal{S}(H)$, let H_S denote a new copy of G_{k-1}. Furthermore, for each stable set $X \in \mathcal{S}(H_S)$, introduce a new vertex $v_{S,X}$ adjacent to all vertices in X but no others. Let us denote by H'_S the graph obtained from H_S resulting from these vertex additions. The graph G_k is then defined as the union of H and H'_S over all $S \in \mathcal{S}(H)$. Its collection $\mathcal{S}(G_k)$ consists of two sets for each $S \in \mathcal{S}(H)$ and $X \in \mathcal{S}(H_S)$, namely: $S \cup X$ and $S \cup \{v_{S,X}\}$. Observe that $S \cup X$ and $S \cup \{v_{S,X}\}$ are both stable sets of G_k.

Burling defined the family $\{G_k\}$ in his PhD Thesis [2] in 1965 and proved that these graphs have unbounded chromatic number. However, this construction went mostly unnoticed until it was rediscovered in [10]. (One exception is a set of unpublished lecture notes of Gyárfás [7] from 2003, which has a section devoted to Burling graphs.)

Theorem 1.1 **([2])** *For every $k \geqslant 1$, the Burling graph G_k is triangle free and has chromatic number at least k.*

[1] This paper is an extended abstract. For the full version containing the proof of the main theorem see arXiv:1703.07871

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