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Towards a problem of Ruskey and Savage on matching extendability ⋆

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Abstract

Does every matching in the n-dimensional hypercube Q_n extend to a Hamiltonian cycle? This question was raised by Ruskey and Savage in 1993 and even though a positive answer in known in some special cases, the problem still remains open in general. In this paper we present recent results on extendability of matchings in hypercubes to Hamiltonian cycles and paths as well as on the computational complexity of these problems, motivated by the Ruskey-Savage question. Moreover, we verify the conjecture of Vandenbussche and West saying that every matching in Q_n, $n \geq 2$, extends to a 2-factor.

Keywords: Hamiltonian cycle, hypercube, matching, Ruskey and Savage problem

1 Introduction

The n-dimensional hypercube Q_n is the graph on the set of all n-bit strings with edges joining two vertices whenever they differ in exactly one bit. There is a large literature on structural properties of this class of graphs which comes from research on the topological structure and analysis of hypercubic interconnection networks [22].

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It is well known that Q_n is Hamiltonian for every $n \geq 2$. Among a number of appealing problems related to Hamiltonicity of hypercubes, the most prominent role was played by the notorious Middle Levels Conjecture, recently resolved by Mütze [15]. But there is another long-standing question, raised in 1993 by Ruskey and Savage [18], asking whether every matching in Q_n extends to a Hamiltonian cycle. A positive solution has been verified for $n \leq 5$ by a computer search [23], but for larger values of n, the answer is known only in several special cases. It may be of interest that matchings in hypercubes that can be avoided by a Hamiltonian cycle were characterized in [2].

The purpose of this paper is to present recent results on extendability of matchings in hypercubes to Hamitonian cycles, Hamiltonian paths and 2-factors as well as on the computational complexity of these problems, motivated by the Ruskey-Savage question.

2 Preliminaries

Throughout this paper, n always denotes an integer such that $n \geq 2$. Given an n-bit string u, we use u_i to denote the i-th element of the sequence $u_1 \dots u_n = u$. Vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively.

Given the n-dimensional hypercube Q_n, the *parity* $\chi(v)$ of a vertex v of Q_n is defined by $\chi(v) = \prod_{i=1}^{n} (-1)^{v_i}$. A set $S \subseteq V(Q_n)$ is called *balanced* if $\sum_{v \in S} \chi(v) = 0$. We use $d(u, v)$ to denote the Hamming distance of $u, v \in V(Q_n)$, i.e. $d(u, v) = |\{i \mid u_i \neq v_i\}|$. The *dimension* of an edge uv of Q_n is defined as the integer i such that $u_i \neq v_i$. If $u_i = 0 \neq v_i$, the parity of an edge uv is defined as the parity of u. The set of all edges of Q_n of the same dimension and the same parity is called a *half-layer*.

Let $K(Q_n)$ denote the complete graph on the set of vertices of Q_n. We use $B(Q_n)$ to denote the spanning subgraph of $K(Q_n)$ containing only edges $uv \in E(K(Q_n))$ such that $d(u, v)$ is odd. While $K(Q_n)$ is a completion of Q_n, $B(Q_n)$ may be viewed as a bipartite completion of Q_n.

A *matching* is a set of edges without common vertices. A matching M in a graph G is called *perfect* if every vertex of G is incident with an edge of M. A matching M in $K(Q_n)$ is called *balanced* if the set of all vertices that are incident with an edge of M forms a balanced subset of $V(Q_n)$. We say that a set of edges P of $K(Q_n)$ is *extendable* if there exists a set of edges R of Q_n such that $P \cup R$ is a Hamiltonian cycle of $K(Q_n)$.