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On packing spanning arborescences with matroid constraint

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Abstract

Let us be given a rooted digraph $D = (V + s, A)$ with a designated root vertex s. Edmonds' seminal result [3] states that D has a packing of k spanning s-arborescences if and only if D has a packing of k (s,t)-paths for all $t \in V$, where a packing means arc-disjoint subgraphs.

Let $\mathcal{M}$ be a matroid on the set of arcs leaving s. A packing of (s,t)-paths is called $\mathcal{M}$-based if their arcs leaving s form a base of $\mathcal{M}$ while a packing of s-arborescences is called $\mathcal{M}$-based if, for all $t \in V$, the packing of (s,t)-paths provided by the arborescences is $\mathcal{M}$-based. Durand de Gevigney, Nguyen and Szigeti proved in [2] that D has an $\mathcal{M}$-based packing of s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s,t)-paths for all $t \in V$. Bérczi and Frank conjectured that this statement can be strengthened in the sense of Edmonds' theorem such that each s-arborescence is required to be spanning. Specifically, they conjectured that D has an $\mathcal{M}$-based packing of spanning s-arborescences if and only if D has an $\mathcal{M}$-based packing of (s,t)-paths for all $t \in V$.

We disprove this conjecture in its general form and we prove that the corresponding decision problem is NP-complete. However, we prove that the conjecture

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holds for several fundamental classes of matroids, such as graphic matroids and transversal matroids.

Keywords: connectivity, packing arborescences, Edmonds' branching theorem, matroid

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1 Introduction

Packing arborescences, or more generally, packing problems concerning connectivity in directed graphs are fundamental subjects in combinatorial optimization. Here, by packing subgraphs in a directed graph, we mean a set of arc-disjoint subgraphs. The question of reachability is one of the basics in the area of connectivity in digraphs. Suppose that we are given a **rooted digraph**, i.e. a digraph $D = (V + s, A)$ with a designated root vertex s. Let S be the set of vertices reachable from s in D. The definition of the reachability says that, for each $t \in S$, D has an (s,t)-path, which certificates that t belongs indeed to S. Now, consider storing such certificates for all vertices in S. Then storing an s-arborescence on S would be the most compact way for keeping all the certificates simultaneously.

To extend this idea to a more general setting, suppose that D has a packing of k (s,t)-paths from s to each vertex t in V, and suppose that we want to provide a certificate that D has indeed such a property. Then the most compact certificate would be to exhibit k arc-disjoint spanning s-arborescences in D. The following fundamental theorem of Edmonds [3] claims that such a compact certificate always exists.

Theorem 1.1 ([3]) *There exists a packing of k spanning s-arborescences in a rooted digraph $D = (V + s, A)$ if and only if there exists a packing of k (s,t)-paths in D for every $t \in V$.* □

The problem of packing k (s,t)-paths is equivalent to asking whether one can send k distinct commodities from s to t by assuming that each arc can transmit at most one commodity. Then what happens if commodities have a more involved independence structure? Here we are interested in a situation that each arc from the root can be used to transmit only a particular commodity, and we would like to know whether every vertex can receive a sufficient

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