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Embedding graphs in Euclidean space

Nóra Frankl [1,3]

Department of Mathematics
London School of Economics
London, UK

Andrey Kupavskii [2,4]

Moscow Institute of Physics and Technology
Moscow, Russia
Ecole Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

Konrad J. Swanepoel [5]

Department of Mathematics
London School of Economics
London, UK

Abstract

The dimension of a graph G is the smallest d for which it can be embedded in $\mathbb{R}^d$ as a unit distance graph. Answering a question of Erdős and Simonovits, we show that any graph with less than $\binom{d+2}{2}$ edges has dimension at most d. Improving their result, we also prove that the dimension of a graph with maximum degree d is at most d.

Keywords: Unit distance graph, Graph representation, Graph dimension

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1 Introduction

Definition 1.1 A graph $G = (V, E)$ is a *unit distance graph* in $\mathbb{R}^d$, if $V \subset \mathbb{R}^d$ and

$$E \subseteq \{(x, y) : x, y \in V, |x - y| = 1\}.$$

Note that we do not require the edge set of a unit distance graph to contain all unit-distance pairs.

We say that a graph G is *realizable* in a subset X of $\mathbb{R}^d$, if there exists a unit distance graph G' in $\mathbb{R}^d$ on a set of vertices $X_0 \subset X$, which is isomorphic to G. We will use this notion for $X = \mathbb{R}^d$ and for $X = \frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$, where $\frac{1}{\sqrt{2}}\mathbb{S}^{d-1}$ is a sphere of radius $1/\sqrt{2}$ with center in the origin.

In the paper [3], Erdős, Harary and Tutte introduced the concept of the Euclidean dimension $\dim G$ of a graph G.

Definition 1.2 The *Euclidean dimension* $\dim G$ (*spherical dimension* $\dim_S G$) of a graph G is equal to k, if k is the smallest integer such that G is realizable in $\mathbb{R}^k$ (on $\frac{1}{\sqrt{2}}\mathbb{S}^{k-1} \subset \mathbb{R}^k$).

They studied the dimension of graphs, e.g, complete graphs, wheels, complete bipartite graphs, cubes. They also study the relation of the dimension to the chromatic number of the graph and to its girth.

In [4] it was shown that if G has maximum degree d then $\dim G \leq \dim_S G \leq d + 2$. We prove something stronger.

Theorem 1.3 *Let $d \geq 2$. Any graph $G = (V, E)$ with maximum degree $d - 1$ has spherical dimension at most d.*

We also prove the following.

Theorem 1.4 *Let $d \geq 1$ and let $G = (V, E)$ be a graph with maximum degree d. Then G is a unit distance graph in $\mathbb{R}^d$ except if $d = 3$ and G contains $K_{3,3}$.*

Definition 1.5 Let $f(d)$ denote the least number for which there is a graph with $f(d)$ edges that is not realizable in $\mathbb{R}^d$.

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[3] Email: n.frankl@lse.ac.uk

[4] Email: kupavskii@ya.ru

[5] Email: k.swanepoel@lse.ac.uk